Summary (for Lec 10)

to ch 9: we have obtained an idea how the GOE and GSE
can solved using skew OP $\sim|\Delta_N|^1$ $\sim|\Delta_N|^4$

e.g. with skew product for $\beta=4$ $\langle f,g\rangle_4 = \int [f(x)g'(x) - f'(x)g(x)]w(x)\,dx$

replacing $\Delta_N^4 = \det_{2N\times 2N}[Q_j(x_i)\ Q_j'(x_i)]_{\substack{i=1..N\\ j=0,\dots,2N-1}}$

and choosing (*) $\langle Q_{2j}, Q_{2k+1}\rangle_4 = h_j^{(4)}\delta_{kj}$, Q_j skew OP, h_j skew norms

$\Rightarrow$ the normalising partition function $Z_N^{(\beta=4)} \sim \prod_{j=0}^{N-1} h_j^{(4)}\, \underbrace{Pf[\langle Q_i, Q_j\rangle_4]}_{\overset{!}{=}1 \ (*)} N!$

given by a Pfaffian determinant simplifies

- likewise the k-point correlation functions can be expressed through Pf-det of size 2k, containing a 2x2 matrix valued kernel of the skew OP Q_j for both $\beta=1$ and 4

to ch 10: Gap probabilities

- thanks to the Andréief formula $\int dx_1..dx_N \det[\phi_i(x_j)]\det[\psi_i(x_j)] = N!\det[\int dx\,\phi_i(x)\psi_j(x)]$
we can compute the gap probab. A_I that $I\subseteq\mathbb{R}$ is void of eigenvalues

as $\boxed{A_I = \det\left[\delta_{ij} - \int_I dx\, w(x) P_i(x) P_j(x)\right]_{i,j=1}^{N}}$, valid for ensembles with $(\Delta_N)^2$

ON polynomials

- remark: in principle this can be used to derive the spacing distribution for the GUE in the bulk of the spectrum for finite-N, see [M.L. Mehta ch 6.2]
  $\rightarrow$ the result is an N-fold product in terms of spheroidal functions, cf. —u—
  for a numerical comparison to the approximate Wigner surmise Fig 1,5

$\rightarrow$ back to the k-th gap probab. defined as $E_k^{(\beta)}(s) = ..$

$$\Rightarrow \quad \frac{\partial}{\partial s} E_k^{(\beta)}(s) = p_k^{(\beta)}(s) - p_{k+1}^{(\beta)}(s)$$

$$\Leftrightarrow \quad \boxed{p_k^{(\beta)}(s) = -\sum_{\ell=0}^{k-1} \frac{\partial}{\partial s} E_\ell^{(\beta)}(s)}$$

- if we know all gaps E_k we know all indiv. eigenvalue distributions p_k ($\Rightarrow$ also all k-point corr. e.g. $R_1(x) = \sum_{k=1}^{N} p_k(x)$)
- normalisation: $\forall_k$ $\boxed{\int_{-\infty}^{\infty} dx\, p_k^{(\beta)}(s) = 1}$

exercise or [arXiv 0311121 App. A with $w(x)$ on $\mathbb{R}_+$ $E_k \to k! E_k$]

we will now compute $E_k^{(\beta)}(s)$ (in two ways):

i) expressed through $R_{N,k}^{(\beta)}$'s in general = expansion / rep. of Fredholm

(ii) expressed through $\langle \prod \det's \rangle$ for $w(x) = x^\alpha e^{-x}\,\Theta(x)$) later

i) idea: $(a-b)^j = \sum_{\ell=0}^{j} (-)^\ell \binom{j}{\ell} a^{j-\ell} b^\ell$

with $a = \int_{-\infty}^{\infty} dx$, $b = \int_{-\infty}^{s} dx \Rightarrow a-b = \int_{s}^{+\infty} dx$

using the symmetry of $\mathcal{P}_N^{(\beta)}$ under permutations of arguments:

$$\Rightarrow E_k^{(\beta)}(s) = \binom{N}{k} \int_{-\infty}^{s} d\lambda_1 \dots d\lambda_k \sum_{\ell=0}^{N-k} (-)^\ell \binom{N-k}{\ell} \left(\int_{-\infty}^{\infty}\right)^{N-k-\ell} \left(\int_{-\infty}^{s}\right)^{\ell} d\lambda_{k+1} \dots d\lambda_N\, \mathcal{P}_N^{(\beta)}$$

$$= \sum_{\ell=0}^{N-k} (-)^\ell \frac{N!}{k!\,\ell!\,(N-k-\ell)!} \int_{-\infty}^{s} d\lambda_1 \dots d\lambda_{k+\ell} \int_{-\infty}^{\infty} d\lambda_{k+\ell+1} \dots d\lambda_N\, \mathcal{P}_N^{(\beta)}$$

$$E_k^{(\beta)}(s) = \sum_{\ell=0}^{N-k} (-)^\ell \frac{1}{k!\,\ell!} \int_{-\infty}^{s} d\lambda_1 \dots d\lambda_{k+\ell}\, R_{N,k+\ell}^{(\beta)}(\lambda_1, \dots, \lambda_{k+\ell})$$

here we define $R_{N,0}^{(\beta)} \equiv 1$ $\left(= \frac{Z_N^{(\beta)}}{Z_N^{(\beta)}}\right)$

e.g. $E_0^{(\beta)}(s) = 1 - \int_{-\infty}^{s} dx_1 R_{N,1}^{(\beta)}(x_1) + \frac{1}{2}\int_{-\infty}^{s} dx_1 dx_2 R_{N,2}^{(\beta)}(x_1,x_2) + \ldots$

- numerically this sum (sometimes) converges very rapidly
- in the limit $N\to\infty$ this is the def. of a Fredholm det., alternatively to p. 25

remark: we may also introduce a generating functional

$$E^{(\beta)}(s;\xi) \equiv \sum_{\ell=0}^{N} (-\xi)^\ell \frac{1}{\ell!} \int_{-\infty}^{s} d\lambda_1 \ldots d\lambda_\ell\, R_{N,\ell}^{(\beta)}(\lambda_1 \ldots \lambda_\ell)$$

$$\Rightarrow \quad E_k^{(\beta)}(s) = \frac{(-)^k}{k!}\frac{\partial^k}{\partial \xi^k} E^{(\beta)}(s;\xi)\Big|_{\xi=1} \quad \text{for } k = 0,1,\ldots,N$$

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ii) let us choose the chiral or Wishart-Laguerre ensembles (N×(N+ν) matrices)

with weight $w_\nu^{(\beta)}(x) = x^{\frac{\beta}{2}(\nu+1)-1}\, e^{-x}$ on $\mathbb{R}_+$ $\Rightarrow$ all lower $\int$-bounds are at 0

consider $E_{0,\nu}^{(\beta)}(s)$: display ν-dep

$$E_{0,\nu}^{(\beta)}(s) = \frac{1}{N!\prod_{j=0}^{N-1} h_j^{(\beta)}} \int_s^\infty \prod_{\ell=1}^{N} dx_\ell\, x_\ell^{\frac{\beta}{2}(\nu+1)-1} e^{-x_\ell} \prod_{n>m}^{N} |x_n - x_m|^\beta$$

shift $x_k - s = y_k$

$$= \frac{1}{N!\prod_{j=0}^{N-1} h_j^{(\beta)}} \int_0^\infty \prod_{\ell=1}^{N} dy_\ell\, (y_\ell+s)^{\frac{\beta}{2}(\nu+1)-1} e^{-(y_\ell+s)} \prod_{n>m} |y_n - y_m|^\beta$$

if we compare to

$$\left\langle \prod_{f=1}^{L} \det(z_f - HH^\dagger) \right\rangle_{N,\nu} = \frac{1}{N!\prod_{j=0}^{N-1} h_j^{(\alpha)}} \int_0^\infty \prod_{\ell=1}^{N} d\lambda_\ell\, \lambda_\ell^{\frac{\beta}{2}(\nu+1)-1} e^{-\lambda_\ell} \prod_{f=1}^{L} (z_f - \lambda_\ell)^{1+\delta_{\beta,4}} |\Delta_N|^\beta$$

11. The Resolvent Method [ch. 8 in LNV]

. consider the following function on $\mathbb{C}$ for a given random matrix $H = H^\dagger$, e.g. from the GUE, with eigenvalues $x_1, \dots, x_N$

Def Resolvent (or Green's function or Stieltjes trafo): For $z \in \mathbb{C} \setminus \{x_1, \dots, x_N\}$

$$\boxed{G_N(z) = \frac{1}{N} \operatorname{Tr} \frac{1}{z - H}} = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{z - x_i} = \int dx \, \frac{1}{z-x} \underbrace{\frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i)}_{\hat{=}\, n(x)}$$

Motivation: we will show that after averaging over H and taking $N \to \infty$,

$$\langle G_N(z) \rangle_N \xrightarrow[N\to\infty]{} G_\infty^{(av)}(z) \underset{\substack{\uparrow \\ \text{plausible from above}}}{=} \int dx \, \frac{n_x(x)}{z - x} \quad , \quad z \in \mathbb{C} \setminus \operatorname{Supp}(n_x)$$

(e.g. $[-\sqrt{2}, \sqrt{2}]$ f. GUE)

1.) satisfies an algebraic eq., i.e. quadratic for GUE rather than a singular integral eq. for $\underset{= \rho(x)}{n_x(x)}$ (5.15): $\underline{x = \mathrm{Pr} \int dx' \, \frac{n_x(x')}{x - x'}}$

2.) from $G_\infty^{(av)}(z)$ the solution for $n_x(x)$ can be obtained, i.e. semicircle ρ_{sc} for GUE

to 2.) uses the Sokhotski–Plemelj–formula

$$\boxed{\lim_{\varepsilon \to 0^+} \frac{1}{y \pm i\varepsilon} = \mathrm{Pr}\left(\frac{1}{y}\right) \mp i\pi\, \delta(y)}$$

i.e. another rep of Dirac-δ

$$\Leftrightarrow \forall \text{ real test functions } \varphi(x): \underbrace{\lim_{\varepsilon \to 0^+} \left[\left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty} \right) dy \, \frac{\varphi(y)}{y} \right]}_{\text{Principal value integral}} \mp \underset{\text{from } \delta}{i\pi \varphi(0)} = \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} dy \, \frac{\varphi(y)}{y \pm i\varepsilon}$$

How does this work for G_∞:

$$G_\infty^{(av)}(z - i\varepsilon) = \int dx \, \frac{\rho(x)}{z - i\varepsilon - x} = \underbrace{\int dx \, \frac{\rho(x)(z-x)}{(z-x)^2 + \varepsilon^2}}_{\xrightarrow[\varepsilon\to 0^+]{} \mathrm{Pr}} + i \int dx \, \rho(x) \underbrace{\frac{\varepsilon}{(z-x)^2 + \varepsilon^2}}_{\xrightarrow[\varepsilon\to 0^+]{} \delta(z-x)\pi} = \underbrace{\mathrm{Pr} \int dx \, \frac{\rho(x)}{z - x}}_{\in \mathbb{R}} + \underbrace{i\pi \rho(z)}_{\in i\mathbb{R}}$$

for $z \in \mathbb{R}$ or better $\operatorname{supp} \rho$

$$\Rightarrow \boxed{\rho(z) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \operatorname{Im} G_\infty^{(a)}(z - i\varepsilon)}$$

inversion formula for $z \in \operatorname{supp} \rho$

to 1.) recall the Coulomb gas picture, Lecture 3, p 15:

$$Z_{N,\beta} = C_{N,\beta} \int_{\mathbb{R}^N} [dx] \exp[-\beta N^2 V[x]]$$, with energy $V[x]$, for Gauss' potential

$$V[x] = \frac{1}{2N}\sum_{i=1}^{N} x_i^2 - \frac{1}{2N^2}\sum_{i\neq j} \ln|x_i - x_j|$$

Now we impose the saddle point (SP) condition before taking the large-N limit

$$\frac{\partial V[x]}{\partial x_i} = 0 \Rightarrow \boxed{x_i = \frac{1}{N}\sum_{\substack{j\neq i \\ j=1}}^{N} \frac{1}{x_i - x_j}} \quad \Big| \cdot \sum_{i=1}^{N}\frac{1}{N(z-x_i)}$$

$$\Rightarrow \frac{1}{N}\sum_{i=1}^{N} \underbrace{\frac{x_i - z + z}{z - x_i}}_{= -1 + \frac{z}{z-x_i}} = \frac{1}{N^2}\sum_{i=1}^{N}\sum_{j\neq i}^{N} \underbrace{\frac{1}{(x_i-x_j)(z-x_i)}}_{= \frac{1}{z-x_j}\left(\frac{1}{z-x_i} - \frac{1}{x_j - x_i}\right)}$$

$$\Rightarrow LHS = -1 + z G_N(z), \qquad \left(G_N'(z) = -\frac{1}{N}\sum_{i=1}^{N}\frac{1}{(z-x_i)^2}\right)$$

Ex

$$\Rightarrow RHS = R = G_N(z)^2 + \frac{1}{N}G_N'(z) - R \Leftrightarrow R = \frac{1}{2}G_N^2(z) + \frac{1}{2N}G_N'(z)$$

$$\Rightarrow \boxed{-1 + z G_N(z) = \frac{1}{2}G_N^2(z) + \underbrace{\frac{1}{2N}G_N'(z)}}$$ SP condition on $G_N(z)$

non-linear diff. eq for $G_N(z)$, But subleading for $N\to\infty$:

as for $x_i = O(1)$ $G_N(z) = O(1)$, e.g. $G_N(z) \underset{|z|\gg 1}{=} \frac{1}{z} + O\left(\frac{1}{z^2}\right)$

$$\underset{N\to\infty}{\overset{\text{ave}}{\Rightarrow}} \boxed{G_\infty^{(av)}(z)^2 - 2z\, G_\infty^{(av)}(z) + 2 = 0}$$ quadratic eq for limiting resolvent

Ex $\Rightarrow G_\infty^{(av)}(z) = z \pm \sqrt{z^2 - 2}$, only "−" satisfies $G_\infty^{(av)}(z) \sim \frac{1}{z}$

* the limiting resolvent has a square root cut on $\mathbb{C}$ along supp S_{sc}.

<u>Ex 10</u>: $\frac{1}{\pi} \operatorname{Im} G_\infty^{(av)}(x - i\varepsilon) \underset{\varepsilon\to 0^+}{\longrightarrow} S_{sc}(x) = \frac{1}{\pi}\sqrt{2-x^2}$