to 1.) recall the Coulomb gas picture, Lecture 3, p 15:

$$Z_{N,\beta} = C_{N,\beta} \int_{\mathbb{R}^N} [dx] \exp[-\beta N^2 V[x]] \quad \text{, with energy } V[x] \text{, for Gauss' potential}$$

$$V[x] = \frac{1}{2N} \sum_{i=1}^{N} x_i^2 - \frac{1}{2N^2} \sum_{i \neq j} \ln |x_i - x_j|$$

Now we impose the saddle point $^{(SP)}$ condition before taking the large-N limit

$$\frac{\partial V[x]}{\partial x_i} = 0 \Rightarrow \boxed{x_i = \frac{1}{N} \sum_{\substack{j \neq i \\ j=1}}^{N} \frac{1}{x_i - x_j}} \qquad \Big| \cdot \sum_{i=1}^{N} \frac{1}{N(z - x_i)}$$

$$\Rightarrow \frac{1}{N} \sum_{i=1}^{N} \underbrace{\frac{x_i - z + z}{z - x_i}}_{= -1 + \frac{z}{z - x_i}} = \frac{1}{N^2} \sum_{i=1}^{N} \sum_{j \neq i}^{N} \underbrace{\frac{1}{(x_i - x_j)(z - x_i)}}_{= \frac{1}{z - x_j} \left(\frac{1}{z - x_i} - \frac{1}{x_j - x_i} \right)}$$

$$\Rightarrow LHS = -1 + z G_N(z), \qquad \left(G_N'(z) = -\frac{1}{N} \sum_{i=1}^{N} \frac{1}{(z - x_i)^2} \right)$$

Ex

$$\Rightarrow RHS = R = G_N(z)^2 + \frac{1}{N} G_N'(z) - R \Leftrightarrow R = \frac{1}{2} G_N^2(z) + \frac{1}{2N} G_N'(z)$$

$$\Rightarrow \boxed{-1 + z G_N(z) = \frac{1}{2} G_N^2(z) + \underbrace{\frac{1}{2N} G_N'(z)}} \qquad \text{SP condition on } G_N(z)$$

non-linear diff. eq for $G_N(z)$, But ↳ subleading for $N \to \infty$:

as for $x_i = O(1)$ $G_N(z) = O(1)$, e.g. $G_N(z) \underset{|z| \gg 1}{=} \frac{1}{z} + O\left(\frac{1}{z^2}\right)$

$$\underset{N \to \infty}{\overset{\text{ave}}{\Rightarrow}} \boxed{G_\infty^{(av)}(z)^2 - 2z G_\infty^{(av)}(z) + 2 = 0} \qquad \text{quadratic eq for limiting resolvent}$$

Ex $$\Rightarrow G_\infty^{(av)}(z) = z - \sqrt{z^2 - 2}, \quad \text{only "}-\text{" satisfies } G_\infty^{(av)}(z) \sim \frac{1}{z}$$

* the limiting resolvent has a square root cut on $\mathbb{C}$ along supp S_{sc}.

Ex 10: $$\frac{1}{\pi} \operatorname{Im} G_\infty^{(av)}(x - i\varepsilon) \underset{\varepsilon \to 0^+}{\longrightarrow} S_{sc}(x) = \frac{1}{\pi} \sqrt{2 - x^2}$$

• How to deal with a more general potential $\frac{1}{2}x^2 \to \vartheta(x)$, to answer questions about universality? Sp. eq. $\frac{\partial V[x]}{\partial x_i} = 0 \Leftrightarrow \vartheta'(x_i) = \frac{1}{N}\sum_{j\neq i} \frac{1}{x_i - x_j}$

11.1. Loop equations for unitary ensembles ($\beta = 2$):

From now on we consider the averaged resolvent $G(z)$:

$$G(z) = \langle G_N(z) \rangle_N = \frac{1}{N}\left\langle \mathrm{Tr}\,\frac{1}{z-H}\right\rangle_N \overset{\text{Villemann series}}{=} \frac{1}{N}\sum_{k=0}^{\infty} \frac{\langle \mathrm{Tr}\, H^k \rangle}{z^{k+1}} \equiv \text{moment generating function } O(1)$$

• let us also define the connected 2-point resolvent (2nd k-point as)

$$G(z_1, z_2) \overset{\text{no factor of } N}{=} \left\langle \mathrm{Tr}\,\frac{1}{z_1 - H}\,\mathrm{Tr}\,\frac{1}{z_2 - H}\right\rangle_{N,\text{connected}} = \left\langle \mathrm{Tr}\frac{1}{z_1-H}\,\mathrm{Tr}\frac{1}{z_2-H}\right\rangle_N - \left\langle \mathrm{Tr}\frac{1}{z_1-H}\right\rangle_N \left\langle \mathrm{Tr}\frac{1}{z_2-H}\right\rangle_N$$

is also of $O(1)$. $\qquad \sim \frac{N}{z_1}\frac{N}{z_2} - \frac{N}{z_1}\frac{N}{z_2}$ cancel, $O(1)$ is zero

(inversion formula $\Rightarrow R_2(x_1,x_2) - R_1(x_1)R_1(x_2)$ "connected")$^{z_1, z_2 \gg 1}$

Derivation of the loop eq. of the unitary ensembles

partition function $Z_N = \int [dH]\, e^{-N\,\mathrm{Tr}\,\vartheta(H)}$

* integral over all indep. matrix elements of $H = H^\dagger$, $H \in \mathbb{C}^{N\times N}$

with $\vartheta(H) = \sum_{j=1}^{\infty} \frac{g_j}{j} H^j$ general potential (we may set some $g_j \equiv 0$ later) (assume analytic)

* the partition function is invariant under redefinitions $H \to H + \epsilon H^k$, $k = 0, 1, \ldots$

i.e. $\left.\frac{dZ_N}{d\varepsilon}\right|_{\varepsilon=0} = 0 \Rightarrow$ use a particular choice of redefinition:

$$H \to H + \varepsilon \sum_{k=1}^{\infty} \frac{H^k}{z^{k+1}} = H + \varepsilon \frac{1}{z-H}$$ to generate an eq. for $G(z)$:

$$\Rightarrow \left| \begin{array}{l} \mathrm{Tr}\,\vartheta(H) \to \mathrm{Tr}\,\vartheta(H) + \varepsilon\,\mathrm{Tr}\left(\frac{1}{z-H}\vartheta'(H)\right) + O(\varepsilon^2) \\ [dH] \to [dH]\left(1 + \varepsilon\left(\mathrm{Tr}\,\frac{1}{z-H}\right)^2\right) + O(\varepsilon^2) \end{array} \right|$$

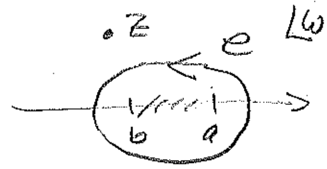
Ex: derive these

$$\Rightarrow 0 = \frac{1}{Z_N}\frac{dZ_N}{d\varepsilon}\Big|_{\varepsilon=0} = \frac{1}{Z_N}\int [dH]\, e^{-N \,\mathrm{Tr}\, v(H)} \left(-N\, \mathrm{Tr}\left(\frac{1}{z-H}\, v'(H)\right) + \left(\mathrm{Tr}\,\frac{1}{z-H}\right)^2\right)$$

$$\Big|\cdot\frac{1}{N^2} \qquad \frac{1}{N}\left\langle \mathrm{Tr}\,\frac{1}{z-H}\, v'(H)\right\rangle_N = \frac{1}{N^2}\left\langle \left(\mathrm{Tr}\,\frac{1}{z-H}\right)^2\right\rangle_N$$, rewrite in terms of $G(z)$,

RHS: $G(z)^2 + \frac{1}{N^2}\, G(z,z)$ well def for $z \in \mathbb{C}\setminus\mathbb{R}$

LHS: We now assume that $N \gg 1$, $n(x) \to n_\infty(x)$ $(\equiv \rho(x))$ with $\sigma = \mathrm{supp}\, n_\infty(x)$ $[b,a]$ ∪ & assuming $z \in \mathbb{C}\setminus\sigma$ (SP: makes sense for $v(H)$ convex)



$$\Rightarrow \; = \int_\sigma dx\, \rho(x)\, \frac{v'(x)}{z-x} = \int_\sigma dx\, \rho(x) \oint_{\mathcal{C}} \frac{dw}{2\pi i}\, \frac{1}{w-x}\, \frac{v'(w)}{z-w}$$

assume ∫'s interchange

$$= \oint_{\mathcal{C}} \frac{dw}{2\pi i}\, \frac{v'(w)}{z-w} \underbrace{\int_\sigma dx\, \frac{\rho(x)}{w-x}}_{=\, G(w)}$$

$$\Rightarrow \text{Loop eq.} \qquad \boxed{\oint_{\mathcal{C}} \frac{dw}{2\pi i}\, \frac{v'(w)}{z-w}\, G(w) = G(z)^2 + \frac{1}{N^2}\, G(z,z)}$$

* a systematic genus expansion of $G(z) = \sum_{g=0}^{\infty} \frac{1}{N^{2g}}\, G_g(z)$ in powers of $\frac{1}{N^2}$ can be made: [Ambjørn et al hep-th/9302014, which we follow closely; cf. Borot, Guionnet 1107.1167 [math.PR] which is rigorous]

11.2. The genus zero $g=0$ (planar) solution $\qquad G_0(z) = G_\infty^{(av)}(z)$

* neglecting the subleading $\frac{1}{N^2}\, G(z,z)$ we have

$$\oint_{\mathcal{C}} \frac{dw}{2\pi i}\, \frac{v'(w)}{z-w}\, G_0(w) = G_0(z)^2$$, deforming the contour $\mathcal{C}$ to $\mathcal{C}_\infty$ we pick up the pole at z, see picture above

$$\Leftrightarrow \boxed{0 = G_0^2(z) - v'(z)\, G_0(z) - \underbrace{\oint_{\mathcal{C}_\infty} \frac{dw}{2\pi i}\, \frac{v'(w)}{z-w}\, G_0(w)}_{=\, Q(z)}} = \text{"quadratic eq."}$$

⇒ formal solution $G_0(z) = \frac{1}{2}V'(z) \pm \frac{1}{2}\sqrt{V'(z)^2 + 4Q(z)}$

with the sign to be fixed from $z \gg 1$: $G_0(z) \sim \frac{1}{z}$ ⇒ "−"

• the first part $V'(z)$ is analytic ⇒ the inversion formula that yields $\rho(x)$ tells us that this contribution comes from $\sqrt{\ }$

⇒ self-consistent Ansatz:

(Δ) $$\boxed{G_0(z) = \frac{1}{2}\left(V'(z) - M(z)\sqrt{(z-a)(z-b)}\right)}$$ recalling $\text{supp}\,\rho = \sigma = [b,a]$

with $M(z)$ to be determined; analytic, so that remaining $\sqrt{\ }$ discont. yields ρ

⇔ $$M(z) = \frac{V'(z) - 2G_0(z)}{\sqrt{(z-a)(z-b)}}, \quad \text{and} \quad M(z) = \oint_{C_\infty} \frac{dw}{2\pi i}\frac{M(w)}{w-z}$$

⇒ $$M(z) = \int_{C_\infty} \frac{dw}{2\pi i}\frac{V'(w) - \cancel{2G_0(w)}}{(w-z)\sqrt{(w-a)(w-b)}}$$ doesn't contribute as $\sim \frac{1}{w}$ at ∞ and $\oint_{C_\infty} \frac{dw}{w^3} = 0$

⇒ $$\boxed{G_0(z) = \frac{1}{2}V'(z) - \frac{1}{2}\int_{C_\infty}\frac{dw}{2\pi i}\frac{V'(w)}{w-z}\sqrt{\frac{(z-a)(z-b)}{(w-a)(w-b)}}} = \frac{1}{2}\oint_{C}\frac{dw}{2\pi i}\frac{V'(w)}{w-z}\sqrt{\frac{(z-a)(z-b)}{(w-a)(w-b)}}$$

original contour

closed form solution for $G_0(z)$ for general potential $V(H)$ for 1 interval supp.

* we still need to determine endpoints a and b in terms of $V(H)$ (or $\{g_i\}$)

These follow from (Δ) by imposing that $G_0(z) = \frac{1}{z} + \mathcal{O}\left(\frac{1}{z^2}\right)$

Exercise: • recover the semi-circle from (Δ) for $V(x) = \frac{1}{2}g_2 x^2$ (Gaussian) what is $a = -b$ in terms of g_2?

• try as well $V(x) = \frac{1}{2}g_2x^2 + \frac{1}{4}g_4x^4$ to determine $G_0(z)$

Can you see when our Ansatz for $\sigma = [b,a]$ would fail?

Universality on a global scale:

- from the inversion formula it follows that

$$\rho(x) = \frac{1}{2\pi} M(x)\sqrt{(a-x)(x-b)}$$

. If $v(x)$ is of degree d then $M(x)$ is a polynomial of degree $d-2$ (why?) and clearly different from the semi-circle $\Rightarrow$ the global macroscopic density in invariant ensembles (cf. Lec. 3) is not universal!

- In the paper by Ambjørn et al. it was shown that the leading order connected 2-point resolvent is universal in the following sense:

$$G_0(p,q) = \frac{1}{4(p-q)^2}\left(\frac{(p-a)(q-b)+(p-b)(q-a)}{\sqrt{(p-a)(p-b)(q-a)(q-b)}} - 2\right)$$

It is the same for all analytic potentials v that share the same endpoints of support $[b,a]$!

For higher order corrections and higher k-point see also there, where an algebraic construction of the genus expansion is made

* genus: from map of unitary ensemble to triangulation of random surfaces ($\to$ application to 2d Quantum gravity)

etc.

genus $g=0$ sphere — $g=1$ torus

"planar" contribution — $\frac{1}{N^2}$ correction, see review

arXiv: hep-th/9306153 P. Di Francesco, P. Ginsparg, J. Zinn-Justin