12. The Wishart – Laguerre Ensembles (= chiral ~) [ch 13 LNV] [& ch 14]

Names: John Wishart paper in Biometrika 1928, has Laguerre polynomials as OP and has chiral symmetry.

* here we consider H to be an $N \times M$ matrix without further symmetry (and real, complex or quaternion elements: $\beta = 1,2,4$)
without loss of generality we consider $M \geq N$ ($M < N$: analog.)

e.g. $H = \begin{pmatrix} H_{11} & H_{12} & H_{13} \\ H_{21} & H_{22} & H_{23} \end{pmatrix}$ 2×3. ← experiment 1, at $t = t_1, t_2, t_3$ ← " 2, — " —

Motivation: H_{ij} from data, e.g. from time series (Wishart: harvest data, Finance → company 1, 2, 3 at day 1, 2, 3... → portfolio optimisation)

→ construct Wishart matrix $\boxed{W = HH^\dagger}$, is $N \times N$, $W = W^\dagger$

- W has N nonnegative eigenvalues $x_1, .., x_N \geq 0$
- Linear algebra: for $H \neq H^\dagger$ ∃ singular value decomp
  $\boxed{H = USV}$, e.g. above $U_{2\times2} \begin{pmatrix} s_1 & 0 & 0 \\ 0 & s_2 & 0 \end{pmatrix} V_{3\times3}$, $U \in U(2), V \in V(3)$
  $\Rightarrow$ $x_i = s_i^2$ $\forall i = 1..N$ squared singular values $s_i \geq 0$

- Start Gauss' distribution for H, change variables to $W \geq 0$

$$Z_{N,\beta} = \int [dH]\, e^{-\frac{\beta}{2} N \mathrm{Tr}\, HH^\dagger} \sim \int [dW]\, (\det W)^{\frac{\beta}{2}(M-N+1)-1} e^{-\mathrm{Tr}\, W \cdot \frac{\beta}{2} N}$$

$\Longrightarrow$ jpdf $\left| \; P(x_1, .., x_N) = \text{const.} \prod_{i=1}^{N} \left(x_i^{\frac{\alpha \beta}{2}} e^{-\frac{\beta}{2} N x_i} \right) \left| \Delta_N(\{x\}) \right|^{\beta} \right.$ on $\mathbb{R}_+^N$, $\alpha = M - N + 1 - \frac{2}{\beta}$, $\beta = 1,2,4$

Ex: derive directly from $dH dH^\dagger$ for $\beta = 2$

goals: 1) find global density via resolvent method
2) find distribution of smallest eigenvalue distribution & local density using OP (exact for finite-N!)

12.1 Saddle point eq. and its solution

$$Z_{M,\beta} = \text{Const} \int_{\mathbb{R}_+^N} dx_1 \dots dx_N \exp[-\beta N^2 V[\{x\}]]$$

with $V[\{x\}] = \frac{1}{2N}\sum_{i=1}^{N}(x_i - \frac{\alpha}{N}\ln x_i) - \frac{1}{2N^2}\sum_{i \neq j}\ln|x_i - x_j| \quad \left(-\mu\sum_{i=1}^{N}\ln x_i\right)$

* saddle point condition:

$$0 = N \cdot \frac{\partial V[\{x\}]}{\partial x_i} = \frac{1}{2} - \frac{\alpha}{2N}\frac{1}{x_i} - \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{x_i - x_j} \qquad \forall i = 1,\dots,N$$

$$\Leftrightarrow \boxed{\frac{1}{2} - \frac{1}{2}\left(\frac{M-N+1}{N} - \frac{2}{\beta N}\right)\frac{1}{x_i} = \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{x_i - x_j}}$$

define $\boxed{c = \frac{N}{M} \leq 1}$ as $M \geq N$, keep c fixed when $M, N \to \infty$

$$\Rightarrow \frac{1}{2} + \frac{1}{2}\left(1 - \frac{1}{c}\right)\frac{1}{x_i} = \frac{1}{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{1}{x_i - x_j} \qquad (*)$$

as before the same SP eq. $\forall \beta = 1,2,4$ (we got the semi-circle for GOE/UE/SE)

* Difference: this eq. has to be satisfied under the constraint $x_i \geq 0 \ \forall i$

$\Rightarrow$ introduce a Lagrange-multiplier μ and a penalty function $-\mu\sum_{i=1}^{N}\ln x_i$ in $V[\{x\}]$, as for $x_i \to 0+$ this sets the jpdf to 0.

$\Rightarrow$ add $-\mu\frac{1}{x_i}$ on LHS of $(*)$, $\cdot \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z - x_i}$

$$\Rightarrow \frac{1}{2}\underbrace{\frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-x_i}}_{G_N(z)} + \left(\frac{c-1}{2c} - \mu\right)\frac{1}{N}\sum_{i=1}^{N}\underbrace{\frac{1}{x_i(z-x_i)}}_{=\left(\frac{1}{x_i} + \frac{1}{z-x_i}\right)\frac{1}{z}} = \frac{1}{N^2}\sum_{\substack{i=1\\ j=1\\ j\neq i}}^{N}\frac{1}{(z-x_i)(x_i - x_j)}$$

$$= \left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{x_i} + G_N(z)\right)\frac{1}{z}$$

$\to \int dx \frac{\rho(x)}{x} \equiv K$ const, may depend on c, μ!

· we will send $\mu \to 0$ at the end of the calculation

$\Rightarrow$ in the large-N limit we obtain for $G_N(z)$ (see p. 57, lec 12)

$$\underline{\frac{1}{z}G_N(z) + \left(\frac{c-1}{2c} - \mu\right)\frac{1}{z}\left(K + G_N(z)\right) = \frac{1}{2}G_N^2(z) + \frac{1}{2N}G_N'(z)}$$

<u>Remark</u>: Had we not introduced a constraint, that is we would have $\mu = 0$, <u>naively setting</u> $c = 1$ ($M = N$, quadratic case) leads to $G_{av}^{(\infty)}(z) = G_{av}^{(\infty)}(z)^2$, with a solution either 0 or 1, not meaningfull! However, below we will find $K \sim \frac{1}{1-c}$.

set $\boxed{\rho = \frac{1-c}{c}}$ to solve the quadratic eq.

$$G_{av}^{(\infty)}(z)^2 - G_{av}^{(\infty)}(z) + 2\left(\frac{\rho}{2}+\mu\right)\frac{G_{av}^{(\infty)}(z)}{z} + 2\left(\frac{\rho}{2}+\mu\right)\frac{K}{z} = 0$$

$$\Rightarrow G_{av}^{(\infty)}(z) = \left(\frac{1}{2} - \frac{\rho}{2z} - \frac{\mu}{z}\right) \pm \sqrt{\left(\frac{1}{2}\left(1-\frac{\rho}{z}\right) - \frac{\mu}{z}\right)^2 - 2\left(\frac{\rho}{2}+\mu\right)\frac{K}{z}}$$

now send $\mu \to 0$

$$\Rightarrow G_{av}^{(\infty)}(z)\Big|_{\mu=0} = \frac{1}{2}\left(1 - \frac{\rho}{z}\right) \pm \sqrt{\frac{1}{4}\left(1 - \frac{2\rho}{z} + \frac{\rho^2}{z^2}\right) - \frac{\rho K}{z}}$$

choose again "$-$" for $G \sim \frac{1}{z}$

$$= \frac{1}{2}\left(1-\frac{\rho}{z}\right) \pm \frac{1}{2z}\sqrt{\underbrace{z^2 - 2\rho z - 4\rho K z + \rho^2}_{\overset{!}{=}(z-z_+)(z-z_-)}} = \frac{1}{2}\left(1-\frac{\rho}{z}\right) - \frac{1}{2z}\sqrt{(z-z_+)(z-z_-)}$$

* to find the limiting support we need to factorise $\overset{!}{=} (z-z_+)(z-z_-)$ (Ex $z_{\pm} = a, b$)

$$\Rightarrow \underline{z_{\pm}} = (1+2K)\rho \pm \sqrt{\rho^2(1+2K)^2 - \rho^2} = \rho\left((1+2K) \pm \sqrt{4K^2+4K}\right) = \rho\left(1+2K \pm 2\sqrt{K^2+K}\right)$$

* this still depends on K, we need to make sure that the resulting density

$$\boxed{\rho_{MP}(z) = \frac{1}{\pi}\lim_{\varepsilon\to 0^+} \mathrm{Im}\, G_{av}^{(\infty)}(z - i\varepsilon) = \frac{1}{2\pi z}\sqrt{(z_+-z)(z-z_-)}}$$ satisfies

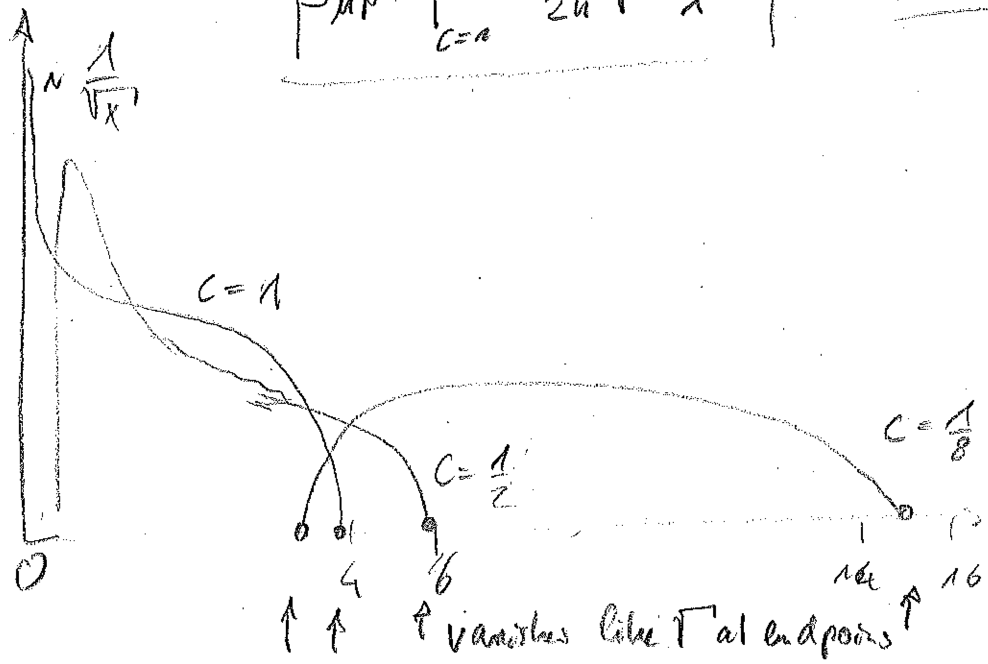
<u>Ex</u>: show that
$$\left.\begin{aligned} &\int_{z_-}^{z_+} dx\, \rho_{MP}(x) = \int_{z_-}^{z_+} dx\, \frac{\sqrt{(z_+-x)(x-z_-)}}{2\pi x} \overset{!}{=} 1 \\ \text{and } &\int_{z_-}^{z_+} dx\, \frac{\rho_{MP}(x)}{x} = K \end{aligned}\right\} \Rightarrow K = \frac{1}{\rho} \text{ is consistent}$$

the insertion of $k = \frac{1}{\beta} = \frac{c}{1-c}$ leads to the simple answers for

Ex show this $\quad z_\pm \overset{!}{=} \left(\frac{1}{\sqrt{c}} \pm 1\right)^2$; recalling that $0 < c = \lim_{N,M\to\infty} \frac{N}{M} \leq 1$

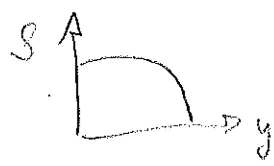
with $\boxed{\rho_{MP}(x) = \frac{1}{2\pi x}\sqrt{(z_+ - x)(x - z_-)}}$ Marchenko-Pastur density

* note that for $c \to 1$ we have $z_- \to 1$, $z_+ \to 4$ and thus the simpler result $\boxed{\rho_{MP}(x)\Big|_{c=1} = \frac{1}{2\pi}\sqrt{\frac{(4-x)}{x}}}$ "quarter circle law"



* Why quarter circle?

$$1 = \int_0^4 dx \frac{1}{2\pi}\sqrt{\frac{4-x}{x}} \underset{\substack{x=y^2 \\ dx = 2dy\,y}}{=} \int_0^2 dy\, y\, \frac{1}{\pi}\sqrt{\frac{4-y^2}{y^2}} \quad , \quad \rho(y) = \frac{1}{\pi}\sqrt{4-y^2} \text{ on } [0,2]$$



* quantitative behaviour:
recall that W has $M-N$ zero eigenvalues.

- for $M - N = O(1)$ a finite # of zero: $\frac{M-N}{N} = \frac{1}{c} - 1 = O\left(\frac{1}{N}\right) \to 0$
  $\Rightarrow$ we are in the case $c = 1$
- for $M - N = O(N)$: $\frac{M-N}{N} = \underbrace{\frac{1}{c} - 1}_{>0} = O(1)$, $\Rightarrow$ case $0 < c < 1$

$\Rightarrow$ because the zero eigenvalues repell the other eigenvalues from the origin, $P_{jpdf} \sim \prod x_i^{\alpha\beta}$ vanishes when $x_j \to 0$, and we have $O(N)$ zero eigenvalues (many!).

the global density gets pushed away from the origin by a "macroscopic" amount: the gap to the origin $z_- = (\sqrt{c}^{-1} - 1)$ is of $O(1)$.

12.2 Chiral symmetry and an application to Quantum Chromodynamics (QCD) [cf. 1103.06011]

- in physics one distinguishes Bosons (int. spin $= 0, 1 ..$ e.g. photon) and Fermions (half-integer spin $= \frac{1}{2}, \frac{3}{2} ...$ e.g. Electron, Quark)
- in a relativistic quantum theory (invariant under Lorentz-trafos) Fermions typically satisfy the Dirac equation

$$(\underbrace{\not\partial}_{} + \underset{\text{mass}}{m})\,\psi = E\,\underset{\text{4 vector}}{\psi}\ ,\qquad \not\partial = \sum_{\mu=1..4} \gamma_\mu \underset{\substack{\text{partial der.}\\ \text{wrt space time}}}{\partial_\mu}$$

Dirac op
4x4 matrix valued

where γ_μ, Dirac matrices, satisfy $\underbrace{\{\gamma_\mu, \gamma_\nu\} \equiv \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}}_{\text{Clifford algebra}}$
(here in Euclidean space-time)

- Chiral symmetry: def $\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}$ in standard rep.

it holds $P_\pm = \frac{1}{2}(\mathbb{1}_4 \pm \gamma_5)$ forms a projection operator

$$P_\pm^2 = P_\pm\ ,\quad P_+ P_- = P_- P_+ = 0\ ,\quad P_+ + P_- = \mathbb{1}_4$$

* in QCD the Dirac op contains a (gauge-field) connection

$$\not{D} = \sum_{\mu=1}^{4} \gamma_\mu \left(\partial_\mu + i g A_\mu(x)\right) \qquad [D_\mu, D_\nu] = -i g_s F_{\mu\nu}$$

↑ connection, with field strength $F_{\mu\nu}$

chiral symmetry: $0 = \{\not{D}, \gamma_5\}$
+ anti hermiticity $\not{D}^\dagger = -\not{D}$

$$\left.\right\} \quad \not{D} = \begin{pmatrix} 0 & i\mathcal{H} \\ i\mathcal{H}^\dagger & 0 \end{pmatrix}$$

block structure

Consequence: $\not{D}\psi_k = i\lambda_k \psi_k \underset{\{\not{D},\gamma_5\}=0}{\overset{\cdot\gamma_5}{\Rightarrow}} \not{D}\gamma_5\psi_k = -i\lambda_k \gamma_5 \psi_k$

purely imaginary eigenvalues come in pairs

$$Z = \int [dA][d\psi] \exp\left[-\int d^4x \left(\sum_f \bar\psi_f(\not{D}+m)\psi_f\right) + \frac{1}{4} \mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu}\right]$$

replace by a simpler random matrix model, vast approx:

$$Z_{RMT} = \int [dH] \det[\not{D}_{RMT} + m]^{N_f} e^{-\mathrm{Tr}\, HH^\dagger}, \quad \text{where } \mathcal{H} \to H$$

eigenvalues of $D_{RMT} = \begin{pmatrix} 0 & iH \\ iH^\dagger & 0 \end{pmatrix}$

$$0 = \det(\lambda - D_{RMT}) = \det\begin{pmatrix} \lambda & -iH \\ -iH^\dagger & \lambda \end{pmatrix} = \det(\lambda^2 - HH^\dagger)$$

$= \pm$ Singular values of H, $\quad \det\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \det(D - BA^{-1}C)$
$\exists A^{-1}$