12.3. Solution using Laguerre polynomials

- From exercise 12. we learned about the jpdf of ev of W $x_1, \dots, x_N$:

$$\underline{P_N^\alpha(x_1, \dots, x_N) = \overbrace{\frac{1}{Z_N^\alpha}}^{\text{Const}} \prod_{i=1}^N x_i^\alpha e^{-x_i} \Delta_N(\{x\})^2} \qquad \beta = 2$$

removed N. (e.g. by rescaling)

where $\alpha = M - N$.

Lec 7, Chapter 8 p 35: solution $\forall$ k-point correl function using OP

we seek $\tilde{P}_k(x) = x^k + \dots$ s.th. $\left| \int_0^\infty dx\, x^\alpha e^{-x} \tilde{P}_k(x) \tilde{P}_\ell(x) = \delta_{k,\ell} h_k \right\} \forall k,\ell = 0,1,\dots$

Solution: classical gen. Laguerre polynomials $L_n^\alpha(x)$ satisfy

$$\int_0^\infty dx\, x^\alpha e^{-x} L_n^\alpha(x) L_m^\alpha(x) = \delta_{n,m} \frac{\Gamma(n+\alpha+1)}{n!} \qquad \text{/ see NIST}$$

with $L_n^\alpha(x) = \sum_{m=0}^n \frac{(-1)^m}{m!} \frac{\Gamma(n+\alpha+1)}{\Gamma(n-m+1)\Gamma(m+\alpha+1)} x^m = \frac{(-)^n}{n!} x^n + \dots$

$$\Rightarrow \boxed{\tilde{P}_n(x) = (-)^n n!\, L_n^\alpha(x)}. \qquad \Rightarrow h_n = \Gamma(n+\alpha+1)\, n!$$

Here, we can generalise to $\alpha \in \mathbb{R}$ with $\boxed{\alpha > -1}$ (for integrals to exist (matrix rep: $P(\tilde{H}) = \det(\tilde{H}\tilde{H}^\dagger)^\alpha e^{-\mathrm{Tr}\tilde{H}\tilde{H}^\dagger}$, $\tilde{H}$ $N \times N$)

$$\Rightarrow \underline{\text{Kernel}} \boxed{K_N^\alpha(x,y) = \sum_{n=0}^{N-1} \frac{n!}{\Gamma(n+\alpha+1)} L_n^\alpha(x) L_n^\alpha(y)} \quad \text{of } \underline{\text{orthonormal polynomials}}$$

p. 37: normalisation constant = partition function

$$\boxed{Z_N^\alpha} = \int_{\mathbb{R}_+^N} dx_1 \dots dx_N \prod_{i=1}^N x_i^\alpha e^{-x_i} \Delta_N(\{x\})^2 = N! \prod_{j=0}^{N-1} h_j = N! \prod_{j=0}^{N-1} j!\, \Gamma(j+\alpha+1)$$

* all k-point correlation functions:

$$R_{N,k}(x_1,\dots,x_k) \equiv \frac{N!}{(N-k)!}\int_0^\infty dx_{k+1}\cdots\int_0^\infty dx_N\, P_N(x_1,\dots,x_N)$$

$$\overset{\text{Lec 3}}{\underset{\text{p 44}}{=}} \prod_{l=1}^{k} x_l^{\alpha} e^{-x_l} \det_{k\times k}\left[K_N^{\alpha}(x_i,x_j)\right]_{i,j=1}^{k}$$

* example spectral density $R_{N,1}(x_1)$:

orthonormal

p. 45 $\quad R_{N,1}(x) = x^{\alpha}e^{-x}K_N^{\alpha}(x,x) = x^{\alpha}e^{-x}\, c_N \left(P_N'(x)P_{N-1}(x) - P_N(x)P_{N-1}'(x)\right)$

$$c_N = \sqrt{\frac{h_N}{h_{N-1}}}$$

where $P_n(x) = \dfrac{\tilde{p}_n(x)}{\sqrt{h_n}} = \dfrac{\sqrt{n!}\,(-)^n}{\sqrt{\Gamma(n+\alpha+1)}}\, L_n^{\alpha}(x)$

NIST $\Rightarrow$ $x\dfrac{d}{dx}L_n^{\alpha}(x) = n L_n^{\alpha}(x) - (n+\alpha)L_{n-1}^{\alpha}(x)$

after some algebra (Exercise, plot)

$$\boxed{R_{N,1}(x) = \frac{N!}{\Gamma(N+\alpha)}\, x^{\alpha-1}e^{-x}\left(-L_N^{\alpha}(x)L_{N-1}^{\alpha}(x) + (N+\alpha)L_{N-1}^{\alpha}(x)^2 - (N+\alpha-1)L_N^{\alpha}(x)L_{N-2}^{\alpha}(x)\right)}$$

* example gap probability and distribution of smallest eigenvalue

probability that all eigenvalues are $\geq s$:

$$E_0^{\alpha}(s) = \int_s^\infty dx_1\cdots\int_s^\infty dx_N\, \frac{1}{Z_N}\prod_{i=1}^{N} x_i^{\alpha}e^{-x_i}\prod_{j>i}(x_j-x_i)^2$$

$\Rightarrow$ this can be related to the computation of Z_N by a simple shift:

$x_i \to y_i = x_i - s$: Δ_N doesn't change!

can take out $(e^{-s})^N$

$$E_0^{\alpha}(s) = \frac{1}{Z_N^{\alpha}}\int_0^\infty dy_1\cdots\int_0^\infty dy_N \prod_{j=1}^{N}(y_j+s)^{\alpha}e^{-y_j-s}\prod_{j>i}(y_j-y_i)^2$$

this looks like a partition function with inserted characteristic polynomials e.g. $\underline{\alpha = 0}$: $\underline{E_0^{\alpha=0}(s)} = e^{-Ns} \frac{Z_N^{\alpha=0}}{Z_N^{\alpha=0}} = \underline{e^{-Ns}}$

$\underline{\alpha = 1}$: $E_0^{\alpha=1}(s) = \frac{e^{-Ns}}{Z_N^{\alpha=1}} \underbrace{\int_0^\infty dy_1 \dots dy_N \prod_{j=1}^N \overset{(-)(-s-y_j)}{\overbrace{(y_j+s)}} \prod_{j=1}^N e^{-y_j} \Delta_N(\{y\})^2}_{\sim \langle \det(1-H) \rangle_N^{\alpha=0} = \tilde{p}_N(x)}$

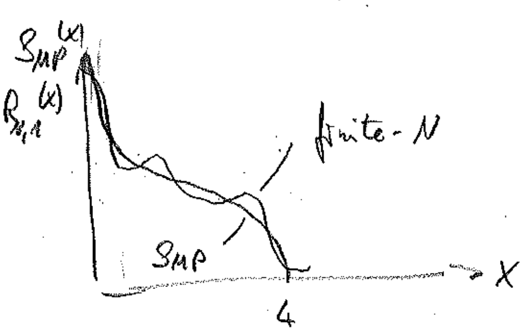
likewise $\Rightarrow \underline{E_0^{\alpha=1}(s) = e^{-Ns} L_N^{\alpha=0}(-s)}$

$\Rightarrow$ distribution of the smallest eigenvalue $\underline{p_1^\alpha(s) = -\frac{\partial}{\partial s} E_0^\alpha(s)}$.

one can show that for $\alpha \in \mathbb{N}$ $E_0^\alpha(s)$ is given by a determinant of size α of Laguerre polynomials

12.4. Large-N limit at the origin and Bessel-kernel

- recall that for $\alpha = 0, 1, 2..$ fixed we have $\rho_{MP}(x) = \frac{1}{\pi}\sqrt{\frac{4-x}{x}}$



- notice that the density at finite-N $\rho_{N,\alpha}$ has no divergence at the origin
- $\exists$ map to quarter circle $x^2 = y$, here the oscillations are visible

like for the GUE (Lec 9, p46) we will magnify the local fluctuations, this time by zooming into the origin:

– for $x = \frac{\hat{x}}{N}$, $\hat{x}$ fixed $x \underset{N\to\infty}{\to} 0$ and the following asymptotic

$$\boxed{\lim_{N\to\infty} N^{-\alpha} L_N^\alpha\left(\frac{\hat{x}}{N}\right) = \hat{x}^{-\frac{\alpha}{2}} J_\alpha(2\sqrt{\hat{x}})},$$

with $J_\alpha(y)$ the Bessel-function of the first kind,

see e.g. NIST handbook

is oscillating

- for the gap probability $E_0^{\alpha=1}\left(\frac{\hat{s}}{N}\right) = e^{-\hat{s}} L_N^{\alpha=0}\left(-\frac{\hat{s}}{N}\right)$

  ↳ minus sign!

we need a different kind of asymptotics

$$\lim_{N\to\infty} N^{-\alpha} L_N^{\alpha}\left(-\frac{\hat{s}}{N}\right) = \underbrace{(-\hat{s})^{-\frac{\alpha}{2}}}_{i^{-\alpha}\hat{s}^{-\frac{\alpha}{2}}} J_\alpha\big(\underbrace{2\sqrt{-\hat{s}}}_{i\sqrt{\hat{s}}}\big) = \hat{s}^{-\frac{\alpha}{2}} I_\alpha\left(2\sqrt{\hat{s}}\right)$$

$\underset{s\gg 1}{\sim} e^{2\sqrt{s}}$ exp growth

as for $\alpha\in\mathbb{N}$ $J_\alpha(iz) = i^\alpha \underline{I_\alpha(z)}$ defines the modified Bessel function I_α of 1st kind

* Results in the quarter circle picture (→ Dirac op. ev)

rescaling $\boxed{x = \frac{\tilde{y}^2}{4N}}$ we obtain the following results in the microscopic origin limit:

- $\alpha=0$: gap probab. $\lim_{\substack{N\to\infty\\ \tilde{y}\text{ fixed}}} E_0^{\alpha=0}\left(\frac{\tilde{y}^2}{4N}\right) = \underline{e^{-\frac{\tilde{y}^2}{4}}}$ $\Rightarrow$ $p_1^{\alpha=0}(\tilde{y}) = -\partial_{\tilde{y}} E_0^{\alpha=0} = \frac{\tilde{y}}{2} e^{-\frac{\tilde{y}^2}{4}}$.

  1st ev

- $\alpha=1$: gap probab. $\lim E_0^{\alpha=1}\left(\frac{\tilde{y}^2}{4N}\right) = \underline{e^{-\frac{\tilde{y}^2}{4}} I_0(\tilde{y})}$

$$\Rightarrow \underline{p_1^{\alpha=1}(\tilde{y})} = -\partial_{\tilde{y}} E_0^{\alpha=1} \underset{\text{exercise}}{=} \frac{\tilde{y}}{2} I_2(\tilde{y}) e^{-\frac{\tilde{y}^2}{4}}$$

1st ev

and one can show for $\alpha\in\mathbb{N}$ $p_1(\tilde{y}) = \frac{\tilde{y}}{2} e^{-\frac{\tilde{y}^2}{4}} \det\left[I_{i-j+2}(\tilde{y})\right]_{i,j=1}^{\alpha}$

- Microsc. and density: using the Christoffel-Darboux formula (Lec 8, p 36)

$$\boxed{K_N(x,y) = c_N \frac{p_N(x)p_{N-1}(y) - p_N(y)p_{N-1}(x)}{x-y}}$$ and its limit $x\to y$

One obtains in the scaling limit above

$$\lim_{N\to\infty} \frac{\tilde{y}}{2N} R_{N,1}\left(\frac{\tilde{y}^2}{4N}\right) = \ldots = \frac{\tilde{y}}{2}\left(J_\alpha(\tilde{y})^2 - J_{\alpha-1}(\tilde{y})J_{\alpha+1}(\tilde{y})\right) = \rho_S(\tilde{y})$$

and likewise for the limiting kernel (unicase)

$$\boxed{K_S(\tilde{y}_1,\tilde{y}_2) = \frac{J_\alpha(\tilde{y}_1)\,\tilde{y}_2\,J_{\alpha-1}(\tilde{y}_2) - J_\alpha(\tilde{y}_2)\,\tilde{y}_1\,J_{\alpha-1}(\tilde{y}_1)}{\tilde{y}_1^2 - \tilde{y}_2^2}} \Rightarrow \text{density}$$

(or an equivalent form with $\alpha-1 \to \alpha+1$)

* because J_α is a limit of the Laguerre OP, which satisfy a 3-term recurrence and simple differential eq., see p.69, also Bessel functions satisfy such identities:

$$\boxed{\begin{aligned} J_\alpha'(x) &= \tfrac{1}{2}\left(J_{\alpha-1}(x) - J_{\alpha+1}(x)\right) \\ 2\alpha J_\alpha(x) &= x\left(J_{\alpha-1}(x) + J_{\alpha+1}(x)\right) \end{aligned}}$$

and similarly for $I_\alpha(x)$

* the density and smallest ev fit beautifully together, see e.g. Fig 3 p.15 in 1603.06011

* zooming into other parts of the spectrum, e.g. in the bulk we obtain the same sine-kernel as in the GUE which is thus universal, using the following asymptotic

$$L_n^\alpha(x) = \frac{n^{\frac{\alpha}{2}-\frac{1}{4}}}{\sqrt{\pi}}\, x^{-\frac{\alpha}{2}-\frac{1}{4}}\, e^{\frac{x}{2}} \cos\left[2\sqrt{nx} - \frac{\alpha\pi}{2} - \frac{\pi}{4}\right] + O\left(n^{\frac{\alpha}{2}-\frac{3}{4}}\right)$$

* deformations of this ensemble to $P(H) \sim e^{-\mathrm{Tr}\,V(HH^+)}$ and Wigner matrices have been studied as well, confirming universality here as well