3.1. Generalisation of the Gaußian Ensembles – a Layman's classification

1. Independent entries : Wigner Ensembles

apart from symmetry requirements, i.e. $H_{ij} \in \mathbb{R}/\mathbb{C}/\mathbb{H}$ and for example $H = H^T$ all matrix elements are independent:

$$\rho(H) \sim \prod_{i=1}^{N} f_i(H_{ii}) \prod_{i<j} f_{ij}(H_{ij})$$

with f_i, f_{ij} not necessarily being Gaußian

Examples: • adjacency matrix of a (random) graph : $H_{ij} = \begin{cases} 1 & \text{nodes } i \text{ and } j \text{ connected} \\ 0 & \text{else} \end{cases}$

⇒ $H = H^T$ symmetry implies that we do not consider a directed graph

• banded random matrix $H =$ 

* in physics the matrix of a Hamiltonian in a finite dim. system is typically not full

2. Invariant ensembles :

the distribution $\rho(H)$ is invariant under conjugation UHU^{-1}:
$= \rho(UHU^{-1})$

e.g. for $H = H^T$ under orthogonal trafos $H' = OHO^T$, $H \in \mathbb{R}^{N\times N}$

$H = H^+$ " unitary " $H' = UHU^{-1}$ $H \in \mathbb{C}^{N\times N}$

$H = \tilde{H}$ symplectic $\mathbb{H}$

⇒ ∃ 2 conditions :

$$\rho(H)\, \underbrace{dH_{11}\, dH_{12} \dots dH_{NN}}_{\text{all indep. matrix el.}} = \rho(H')\, dH'_{11}\, dH'_{12} \dots dH'_{NN}$$

$\Rightarrow$ i) $\rho(H) = \varphi(\mathrm{Tr}\, H, \mathrm{Tr}\, H^2, \dots, \mathrm{Tr}\, H^N)$ Weyl's Lemma

as for H $N \times N$ only the traces of the first N powers are independent

(det H is also invariant, $\det(1-H)$ generates these$\uparrow$, cf. elementary sym. funct.)

ii) $dH_{11} \dots dH_{NN} = dH'_{11} \dots dH'_{NN}$ i.e. the flat Lebesgue measure is invariant under conjugation by U

same structure of jpdf

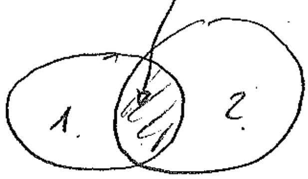
Examples: $\rho_1(H) \sim \exp[-\mathrm{Tr}\, V(H)]$ e.g. V polynomial (potential)

$\rho_2(H) \sim \exp[-(\mathrm{Tr}\, H^2)^2]$ etc

Universality question: for which deformation of the GOE/U/SE do we get the same result, e.g. for $p(s)$?

* What is the intersection of classes 1 & 2?

answ: only the GOE, GUE & GSE



Thm Porter & Rosenzweig, see proof for 2×2 in Fyodorov math-ph/0412017

* ∃ deeper classification in 10 ensembles → Zirnbauer " /0404058

3.2 The spectral density

How can we compute e.g. the probability of a GOE matrix to find an eigenvalue at x, given the jpdf. (2.15) (on p.3)? This should predict Figs 1.1, 3.1.

- take a fixed $N \times N$ matrix H with the real eigenvalues $\lambda_1, \dots, \lambda_N$,

def. counting function $n(x)$ s.th. $\int_a^b dx'\, n(x')$ gives the fraction of eigenvalues of H in $[a,b]$:

we def $\underline{n(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x-x_i)}$

where $\delta(x)$ is the Dirac-distribution

with the property $\int_{\mathcal{I}} dx\, \delta(x-x_0)\, f(x) = \begin{cases} f(x_0) & \text{if } x_0 \in \mathcal{I} \\ 0 & \notin \end{cases}$

for a testfunction $f(x)$.

N.B: a smooth realisation is $\frac{1}{2\sqrt{\pi\varepsilon}}\, e^{-\frac{x^2}{4\varepsilon}} \xrightarrow[\varepsilon\to 0]{} \delta(x)$, for example.

$\Rightarrow$ obviously $n(x)$ does the counting for the fixed matrix H

- if H is a <u>random matrix</u>, e.g. from the ensemble of GOE matrices, $n(x)$ becomes a <u>random measure</u>, and we have to average over the set of rv with jpdf $\rho(x_1,\dots,x_N)$:

$$\underline{\langle n(x) \rangle} := \int [dx]\, \rho(x_1,\dots,x_N)\, n(x) \qquad \text{with } [dx] = dx_1 \dots dx_N$$

$$= \frac{1}{N} \sum_{j=1}^{N} \int [dx]\, \rho(x_1,\dots,x_N)\, \delta(x-x_j)$$

$$= \frac{1}{N} \sum_{j=1}^{N} \int dx_1 \dots dx_{j-1}\, dx_{j+1}\, \rho(x_1,\dots,x_{j-1}, x, x_{j+1},\dots,x_N)$$

$$= \int dx_2 \dots dx_N\, \rho(x, x_2,\dots,x_N), \quad \text{the marginal of the jpdf,}$$

because the jpdf $\rho(x_1,\dots,x_N)$ is invariant (symmetric) under all exchanges $x_j \leftrightarrow x_k$

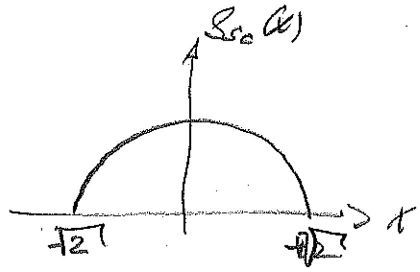
$\boxed{\rho(x) = \langle n(x) \rangle}$ is called the (average) <u>spectral density</u>

also mean or global — " —

Goal: <u>Wigner's semi-circle law</u> $\boxed{\lim_{N\to\infty} \sqrt{\beta N}\, \rho(\sqrt{\beta N}\, x) = \rho_{sc}(x) = \frac{\sqrt{2-x^2}}{\pi}}$

for $\rho(x_1,\dots,x_N) = \frac{1}{Z_{N,\beta}} \prod_{j>k} |x_j - x_k|^{\beta}\, e^{-\frac{1}{2}\sum_{j=1}^{N} x_j^2}$, Gaussian jpdf of β-ensembles

Properties of the semi-circle law



- rescaling:
  for matrices $H = H^\dagger$ with real ($\beta=1$) or complex ($\beta=2$) Gauß' matrix elements $(\mathcal{N}(0, 1/\beta))$ the ev spread over an interval of increasing length $[-\sqrt{2\beta N}, \sqrt{2\beta N}]$

- soft edge:
  in the limit $N \to \infty$ the (rescaled) probab. to find an ev outside $[-\sqrt{2}, \sqrt{2}]$ goes to zero. However, for $N \gg 1$ finite you may find ev there (we'll investigate this later), therefore the density ending as a $\sqrt{\ }$ has a "soft edge".
  In contrast the matrix $HH^\dagger = W$, $H_{ij} \in N(0,1)$ $\forall ij$ is non-negative and has a hard edge at $x=0$: no ev can become negative.

- the rescaling by $\sqrt{\beta N}$ can be done before taking the limit $N \to \infty$:

4. The Coulomb gas picture

- rescale the ev in the jpdf by $x_i \to x_i'$: $x_i = x_i' \sqrt{\beta N}$ $\forall i$:

$$\Rightarrow \underline{Z_{N\beta}} = \int_{\mathbb{R}^N} dx_1 \ldots dx_N \, \varrho(x_1, \ldots, x_N) = \sqrt{\beta N}^N \int_{\mathbb{R}^N} dx_1' \ldots dx_N' \prod_{j<k} |(x_k' - x_j')\sqrt{\beta N}|^\beta e^{-\frac{\beta N}{2}\sum_{i=1}^N x_i'^2}$$

$$= \underline{C_{N,\beta} \int_{\mathbb{R}^N} [dx'] \prod_{j<k}^N |x_k' - x_j'|^\beta \exp\left[-\frac{\beta N}{2} \sum_{i=1}^N x_i'^2\right]}$$

as the Vandermonde-determinant $\prod_{k>j}^N (x_k - x_j)$ contains $\frac{N(N-1)}{2}$ terms (check this)

$$\text{with } C_{N,\beta} = \sqrt{\beta N}^{N + \beta\frac{N(N-1)}{2}}$$

* in the literature you'll find many different conventions $e^{-\frac{N}{2}x^2}$, $e^{-\frac{\beta}{2}x^2}$, e^{-Nx^2} etc. which are all related by rescaling.

⇒ exponentiating the Vandermonde we obtain

$$Z_{N,\beta} = C_{N,\beta} \int_{\mathbb{R}^N} [dx] \, \exp[-\beta N^2 V[x]]$$

with energy $\boxed{V[x] = \frac{1}{2N} \sum_{i=1}^{N} x_i^2 - \frac{1}{2N^2} \sum_{i \neq j} \ln|x_i - x_j|}$

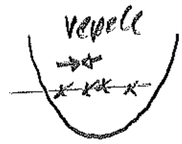
sum over all

Note: the exponent of the Vandermonde = prefactor of ln in the energy <u>cannot</u> be changed by rescaling the ev!

* physical interpretation:

$e^{-\beta N^2 V[x]}$ = Gibbs-Boltzmann weight of a thermodyn. gas (fluid) of N particles at Temperature T, $\beta = \frac{1}{k_B T}$

with <u>logarithmic interaction</u> in a <u>Gaussian potential</u>



<u>Exercise:</u> use <u>Gauss' law</u> for a point charge q, that the (rot. invariant) electric field $E(r)$ integrated over a sphere S in d dimensions $\int_S E \sim q$ yields an electric potential

$$V_d(r) \sim \begin{cases} r & d=1 \\ \ln r & d=2 \\ \frac{1}{r^{d-2}} & d > 2 \end{cases}$$

⇒ ev x_i behave like charged particles in d=2 Coulomb potential confined to the real line, the so called <u>Dyson gas</u>

<u>Convergence:</u> • $\lim_{N\to\infty} V[x]$ seems to exist, $\frac{1}{N}\sum \to \int$

• $N \to \infty$ thermodynamic & zero-temperature limit

⇒ minimize the <u>free energy</u> $\boxed{F = -\frac{1}{\beta} \ln Z_{N,\beta}}$

4.2. Functional integration in the Coulomb gas picture

Goal: minimisation of free energy F by variational principle / functional derivative

Step 1 Start with the counting function $n(x) = \frac{1}{N}\sum_{i=1}^{N} \delta(x-x_i)$,

normalised $\int_{\mathbb{R}} dx\, n(x) = 1$, $n(x) \geq 0$ everywhere on $\mathbb{R}$.

Step 2 coarse graining

integrate over all smoothed random measures that are normalised

$$1 = \int \mathcal{D}[n(x)]\, \delta\Big[n(x) - \frac{1}{N}\sum_{i=1}^{N}\delta(x-x_i)\Big]$$, insert this functional into

$$Z_{N,\beta} = C_{N,\beta}\int \mathcal{D}[n(x)] \int_{\mathbb{R}^N} [dx]\, e^{-\beta N^2 V[x]}\, \delta\Big[n(x) - \frac{1}{N}\sum_{i=1}^{N}\delta(x-x_i)\Big]$$

Step 3 Convert Σ's containing x_i to $\int$ over $n(x)$ in $V[x]$:

$$\sum_{i=1}^{N} f(x_i) = N\int_{\mathbb{R}} dx\, n(x) f(x) , \quad \sum_{i,j=1}^{N} g(x_i,x_j) = N^2\int_{\mathbb{R}^2} dx\,dx'\, n(x)n(x') g(x,x')$$

$\Rightarrow$ problem: in $\sum_{i\neq j} \ln|x_i - x_j|$ the term with $i=j$ diverges logarithmically

introduce a short-distance cutoff $\Delta(x)$, it removes the infinite energy contribution to $V[x]$ when 2 "charges" x_i, x_j become too close, $\sum_{i\neq j} \ln|x_i-x_j| = \sum_{i,j=1}^{N} \ln|x_i-x_j| - \sum_{i=1}^{N}\ln\Delta(x_i)$

$$\Rightarrow V[x] \to V[n(x)] = \frac{1}{2}\int_{\mathbb{R}} dx\, x^2 n(x) - \frac{1}{2}\int_{\mathbb{R}^2} dx\,dx'\, n(x)n(x')\ln|x-x'|$$

(check)

$$+ \frac{1}{2N}\int_{\mathbb{R}} dx\, n(x)\ln\Delta(x)$$

and $$Z_{N,\beta} = C_{N,\beta}\int \mathcal{D}[n(x)]\, e^{-\beta N^2 V[n(x)]} \underbrace{\int_{\mathbb{R}^N} [dx]\, \delta\Big[n(x) - \frac{1}{N}\sum_{i=1}^{N}\delta(x-x_i)\Big]}_{:= I_N[n(x)] \text{ to be evaluated}}$$