4.1. Functional integration in the Coulomb gas picture

Goal: minimisation of free energy F by variational principle / functional derivative

Step 1 Start with the counting function $n(x) = \frac{1}{N}\sum_{i=1}^{N}\delta(x-x_i)$,

normalised $\int_{\mathbb{R}} dx\, n(x) = 1$, $n(x) \geq 0$ everywhere on $\mathbb{R}$.

Step 2 <u>coarse graining</u>

integrate over all <u>smoothed</u> random measures that are normalised

$$1 = \int \mathcal{D}[n(x)]\, \delta\Big[n(x) - \frac{1}{N}\sum_{i=1}^{N}\delta(x-x_i)\Big]$$ insert this functional into

$$Z_{N,\beta} = C_{N,\beta}\int \mathcal{D}[n(x)] \int_{\mathbb{R}^N}[dx]\, e^{-\beta N^2 V[x]}\, \delta\Big[n(x) - \frac{1}{N}\sum_{i=1}^{N}\delta(x-x_i)\Big]$$

Step 3 Convert Σ's containing x_i to $\int$ over $n(x)$ in $V[x]$:

$$\sum_{i=1}^{N} f(x_i) = N\int_{\mathbb{R}} dx\, n(x) f(x), \quad \sum_{i,j=1}^{N} g(x_i,x_j) = N^2\int_{\mathbb{R}^2} dx\,dx'\, n(x)n(x')g(x,x')$$

$\Rightarrow$ problem: in $\sum_{i\neq j}\ln|x_i-x_j|$ the term with $i=j$ diverges logarithmically

introduce a <u>short-distance cutoff</u> $\Delta(x)$, it removes the infinite energy contribution to $V[x]$ when 2 charges x_i, x_j become too close, $\sum_{i\neq j}\ln|x_i-x_j| = \sum_{i,j=1}^{N}\ln|x_i-x_j| - \sum_{i=1}^{N}\ln\Delta(x_i)$

Step 4

$$\Rightarrow V[x] \to V[n(x)] = \frac{1}{2}\int_{\mathbb{R}} dx\, x^2 n(x) - \frac{1}{2}\int_{\mathbb{R}^2} dx\,dx'\, n(x)n(x')\ln|x-x'|$$

(check)

energy functional $$+ \frac{1}{2N}\int_{\mathbb{R}} dx\, n(x)\ln\Delta(x)$$

and $$Z_{N,\beta} = C_{N,\beta}\int \mathcal{D}[n(x)]\, e^{-\beta N^2 V[n(x)]} \underbrace{\int_{\mathbb{R}^N}[dx]\, \delta\Big[n(x) - \frac{1}{N}\sum_{i=1}^{N}\delta(x-x_i)\Big]}_{:= I_N[n(x)] \text{ to be evaluated}}$$

Step 5: Evaluation of $I_N[n(x)]$

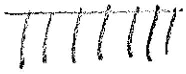
physical interpretation: I_N counts the number of <u>microstates</u> (= config. of $\{x_i\}$) that are compatible with a given <u>macrostate</u> = density profile $n(x)$

Result: $\boxed{I_N[n(x)] \sim \exp\left[-N\int dx\, n(x)\ln n(x)\right]}$

statistical mech. interpretation: <u>Entropy</u>

1. derivation using functional derivatives → exercise 3.2
2. " : combinatorics:

- divide support into K small boxes of equal size
- distribute N particles into K boxes, with <u>occupation numbers</u> $\{n_1, n_2, \dots, n_K\}$, $\sum_{i=1}^{K} n_i = N$



⇒ number of possibilities $\dfrac{N!}{n_1!\, n_2! \dots n_K!} \sim e^{-\sum_i n_i \ln n_i} \sim e^{-N\int dx\, n(x)\ln n(x)}$

using Stirling for $\Gamma(N+1) = N! \sim N^{N+\frac{1}{2}} e^{-N}$

Step 6: what is $\Delta(x)$ = ? – the cut-off or <u>self-energy</u>:

- we have already seen in the derivation of the Poisson $p(\hat{s}) = e^{-\hat{s}}$ spacing distrib. that we had to rescale s by N and the local density $s = \frac{\hat{s}}{N n(x)}$

⇒ Ansatz (in 1 dimension!): $\boxed{\Delta(x) = \dfrac{c}{N n(x)}}$ recalling $n(x)$ is a smooth random measure

where c = const (indep. of N) is undetermined

$$\Rightarrow \int_{\mathbb{R}} dx\, n(x) \ln \Delta(x) = \int_{\mathbb{R}} dx\, n(x)\left(\ln c - \ln N - \ln n(x)\right) \underset{\uparrow}{=} \ln c - \ln N - \int_{\mathbb{R}} dx\, n(x) \ln n(x)$$

normalisation of $n(x)$

⇒ **Step 7.** expand in N

$$Z_{N,\beta} \simeq C_{N,\beta} \int \mathcal{D}[n(x)] \exp\Big[-\beta N^2 \underbrace{\Big(\frac{1}{2}\int dx\, x^2 n(x) - \frac{1}{2}\int dx\, dx'\, n(x) n(x') \ln|x-x'|\Big)}_{=\mathcal{F}_0[n(x)] \text{ energy } O(N^2)} + \frac{\beta}{2} N \ln N$$

$$+ \left(\frac{\beta}{2} - 1\right) N \underbrace{\int dx\, n(x) \ln(n(x))}_{\mathcal{F}_1[n(x)]} - \frac{\beta}{2} N \ln c + O(N) \Big]$$

partition function

self-energy & entropy

* $\beta = 2$ special!

* we have expressed the $\int$ over the jpdf = partition function of a Dyson-Coulomb gas.

→ we still need to show that the limiting density $n_*(x)$ that minimises the free energy is given by the semicircle: $\boxed{n_*(x) \overset{!}{=} \rho_{sc}(x)}$ "equilibrium measure"

⇒ next chapter: saddle point analysis of $Z_{N,\beta}$

Here: compare the result for $Z_{N,\beta=2}$ known from other means (solution of GUE using Hermite polynomials at finite N) for $N \to \infty$ asymptot. large AND what we get by directly inserting $n_* = \rho_{sc}$ at large-N

* $\beta = 2$ GUE ⇒ $\boxed{Z_{N,\beta} = (2\pi)^{\frac{N}{2}} \prod_{i=1}^{N} i!} = (2\pi)^{\frac{N}{2}} G(N+2)$ Barnes G-function

∃ recursion $G(z+1) = \Gamma(z) G(z)$, $G(1) = 1$, see e.g. NIST handbook of Mathematical functions chapter 5.17

- asymptotic of

$$\ln G(z+1) \sim \frac{1}{4} z^2 + z\Gamma(z+1) - \left(\frac{1}{2} z(z+1) + \frac{1}{12}\right) \ln z + O(1)$$

$$\Rightarrow \boxed{\ln Z_{N,\beta=2} \sim \frac{1}{2} N^2 \ln N - \frac{3}{4} N^2 + N \ln N + N(\ln(2\pi) - 1) + \frac{5}{12} \ln N + O(1)}$$

check this!

* On the other hand: inserting $n_*(x) = \frac{1}{\pi}\sqrt{2-x^2}$ in $\mathcal{F}_{0,1}$ yields

$$\mathcal{F}_0[n_*(x)] = \frac{3}{8} + \frac{1}{4} \ln 2, \quad \mathcal{F}_1[n_*(x)] = \frac{1}{2}(1 - \ln 2 - 2\ln\pi)$$ Exercise 3.3

which together with $\ln C_{N,\beta} = \ln(\beta N) \left(\frac{N}{2} + \frac{\beta}{4} N(N-1) \right)$

gives $\boxed{\ln Z_{N\beta} \sim \frac{\beta}{4} N^2 \ln N + a_\beta N^2 + N \ln N + b_\beta N + o(N)}$

Exercise 3.3: determine a_β and b_β

Conclusion(!) the two asymptotic expansions agree for $\beta = 2$ up to including $O(N \ln N)$.

5. Saddle-point analysis of the Coulomb gas partition function

- also called steepest descent method or stationary phase approximation (s.i.) of an integral with a large parameter N

$$I = \int dx\, e^{-Nf(x)} \quad \text{expand around extremum } x_0 \text{ with } f'(x_0) = 0$$

$$\Rightarrow I \approx e^{-Nf(x_0)} \underbrace{\int_{-\infty}^{\infty} dx\, e^{-\frac{Nf''(x_0)}{2}(x-x_0)^2}}_{\text{"fluctuations"}} = \sqrt{\frac{2\pi}{Nf''(x_0)}}\, e^{-Nf(x_0)}$$

see e.g. Fyodorov math-ph/0412017 ch. 5.2.

- for several minima we have to sum over all of them
- for $\int_J$ with $J \subseteq \mathbb{R}$ and min $x_0 \notin J$ we have to expand to linear order at ∂J next to x_0

5.1 Lagrange multiplier and saddle-point equation

in $Z_{N,\beta} = C_{N,\beta} \int \mathcal{D}[n(x)] \exp[-\beta N^2 \mathcal{F}_0[n(x)] + (\frac{\beta}{2}-1) N \mathcal{F}_1[n(x)]] + \frac{\beta}{2} N(\ln N - \ln c) + o(N)]$ (absorb in normalisation)

we have to ensure that the smoothed random measure $n(x)$ is normalised:

insert $\delta[\int_{\mathbb{R}} dx\, n(x) - 1] = \int_{\mathbb{R}} \frac{dk}{2\pi} e^{ik(\int_{\mathbb{R}} dx\, n(x) - 1)}$ using standard δ-rep.

- rescaling $ik \to \beta N^2 \kappa$ and keeping only the leading order $O(N^2)$ $\mathcal{F}_0$-contrib.

$$\Rightarrow Z_{N,\beta} \approx \tilde{C}_{N,\beta} \int \mathcal{D}[n(x)] \int_{\mathbb{R}} d\kappa \exp\{-\beta N^2 \underbrace{[\mathcal{F}_0[n(x)] - \kappa(\int dx\, n(x) - 1)]}_{\equiv S[n(x),\kappa]} + O(N)\}$$

(κ: Lagrange-multiplier)

saddle-point $\approx \text{const}\; e^{-\beta N^2 S[n_*(x), \kappa_*]}$ (we don't consider the fluctuations)

with $n_*(x), \kappa_*$ solving the saddle point equations (SP eq.)

$$\boxed{\begin{aligned} 0 &= \frac{\delta}{\delta n(x)} S[n(x),\kappa]\Big|_{\substack{n=n_*\\ \kappa=\kappa_*}} = \frac{x^2}{2} - \int_{\mathbb{R}} dx'\, n_*(x') \ln|x-x'| - \kappa_* \\ 0 &= \frac{\partial}{\partial \kappa} S[n(x),\kappa]\Big|_{\substack{n=n_*\\ \kappa=\kappa_*}} = \int_{\mathbb{R}} dx\, n_*(x) - 1 \end{aligned}}$$

, using $\frac{\delta n(x)}{\delta n(y)} = \delta(x-y)$ (5.9)

* the normalized solution $n_*(x)$ of the SP eq. (equilibrium measure) cannot be supported on all of $\mathbb{R}$:

for $x \gg 1$ we have $0 = \frac{x^2}{2} - \int_{\mathbb{R}} dx' n_*(x') \ln|x-x'| - k_*$

$$\approx \frac{x^2}{2} - \ln|x| \int_{\mathbb{R}} dx' n_*(x') = \frac{x^2}{2} - \ln|x|$$ which is inconsistent

$\Rightarrow$ the solution $n_*(x) = n_*(x; a, b)$ has compact support on $[a,b] \subseteq \mathbb{R}$

– the parameters a, b have to be determined s.th. $n_*(x; a, b)$ min F!

5.2 Derivative of the SP eq and Tricomi's solution

$\ln|x-x'|$ is not differentiable at $x = x'$

$\to$ weak derivative: for $u(x), v(x) \in \mathcal{L}^1([a,b])$ v is the weak derivative of u

if $\forall\, \varphi(x) \in \mathcal{C}^\infty,\ \varphi(a) = \varphi(b) = 0$: $\int_a^b dx\, u(x) \varphi'(x) = -\int_a^b dx\, v(x) \varphi(x)$

set $u(x) = \int dx' n_*(x') \ln|x-x'|$, we will regularise the singularity at $x=x'$ by ε

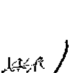
$$\int dx\, \varphi'(x) \int dx' n_*(x') \frac{\ln(x-x')^2}{2} = \frac{1}{2} \lim_{\varepsilon \to 0} \int dx\, \varphi'(x) \int dx' n_*(x') \ln[(x-x')^2 + \varepsilon^2]$$

$$= -\frac{1}{2} \int dx\, \varphi(x) \int dx' n_*(x') \frac{2(x-x')}{(x-x')^2 + \varepsilon^2} = - \int dx\, \varphi(x)\, \underbrace{\mathrm{Pr} \int dx'}_{\text{or } ⨍ dx' \text{ or P.V.} \int dx' \text{ is Cauchy's principal value}} \frac{n_*(x')}{x-x'}$$

i.e. for x singular point of $F(x)$

$$\mathrm{Pr} \int dx' F(x') = \lim_{\varepsilon \to 0} \int^{x-\varepsilon} dx' F(x') + \int_{x+\varepsilon} dx' F(x')$$

$\Rightarrow$ weak derivative of SP eq: $$\boxed{0 = x - \mathrm{Pr} \int dx' \frac{n_*(x')}{x-x'}} \qquad (5.15)$$

Assume that $[a,b]$ is the single interval support of $f(x)$ (why: , not for )

Thm [Tricomi 85]

$$g(x) = \mathrm{Pr} \int_a^b dx' \frac{f(x')}{x-x'} \Rightarrow f(x) = \frac{1}{\pi \sqrt{(x-a)(b-x)}} \Big(C - \mathrm{Pr} \int_a^b \frac{dt}{\pi} \frac{\sqrt{(t-a)(b-t)}}{x-t}\, g(t) \Big)$$

C: arb. (integration) constant