Summary – the bigger picture

- we have introduced several ensembles of random matrices

e.g. the Gaussian Orthogonal Ensemble (GOE)

= probability space on the set of all real $N \times N$ matrices $H \in \mathbb{R}^{N \times N}$ $(M_N(\mathbb{R}))$

that satisfy $H = H^T$ and $\forall i = 1, \dots, N$ H_{ii} distrib according to $\mathcal{N}(0,1)$
$\forall i < j = 1..N$ H_{ij} " " $\mathcal{N}(0, \frac{1}{2})$ } all indep

$$\Leftrightarrow \boxed{P(H) = \underbrace{\frac{1}{2^{\frac{N}{2}} \pi^{\frac{N(N+1)}{4}}}}_{C_N} e^{-\frac{1}{2}\mathrm{Tr}\, H^2}} \quad \text{or} \quad dP(H) = \prod_{i=1}^{N} \frac{e^{-\frac{1}{2}H_{ii}^2}}{\sqrt{2\pi}} dH_{ii} \prod_{i<j} \frac{e^{-H_{ij}^2}}{\sqrt{\pi}} dH_{ij}$$

flat Lebesgue on $\mathbb{R}$

- as the matrices from such an ensemble can be diagonalised, $H = ODO^T$ with $O \in O(N)$, $D = \mathrm{diag}(x_1, \dots, x_N)$ the eigenvalues of H (and $\mathrm{Tr}\, H^2 = \mathrm{Tr}\, D^2$),

we obtain the jpdf $\boxed{\rho(x_1, \dots, x_N) = \frac{1}{Z_{N,\beta}} \prod_{j<k} |x_k - x_j|^\beta e^{-\frac{1}{2}\sum_{i=1}^N x_i^2}}$

normalisation

we still need to prove this, only $N = 2$

with GOE $\beta = 1$, GUE ($H = H^*$, $H \in \mathbb{C}^{N \times N}$, indep (complex) normal)

- Coulomb gas picture (rescale $x_i \to \sqrt{\beta N}\, x_i$)

$$Z_{N,\beta} = C_{N\beta} \int_{\mathbb{R}^N} \underbrace{dx_1 \dots dx_N}_{[dx]} \exp\Big[\underbrace{-\beta N \sum_{i=1}^{N} x_i^2 + \frac{\beta}{2} \sum_{i \neq j}^{N} \ln |x_j - x_i|}_{\equiv -\beta N^2 V[x]}\Big]$$

Gibbs-Boltzmann weight

- Counting function = empirical spectral measure

$n_N(x) = \frac{1}{N} \sum_{i=1}^{N} \delta(x - x_i)$ claim: for $\lim_{N \to \infty} n_N(x) = n_*(x)$ minimises

$$\mathcal{F}_0[n(x)] = \frac{1}{2} \int dx\, x^2 n(x) - \frac{1}{2} \iint dx\, dx'\, n(x) n(x') \ln |x - x'|$$

energy functional

with $Z_{N,\beta} \approx \exp(-\beta N^2 \mathcal{F}_0[n_*(x)])$

where we also computed lower orders $+ (\frac{\beta}{2} - 1) N \overbrace{\int dx\, n(x) \ln n(x)}^{\mathcal{F}_1[n(x)]}$ etc.

- Check: $Z_{N,\beta=2}$ known exactly (to be derived) agrees asymptotically with result from inserting $n_*(x) = \frac{1}{\pi}\sqrt{2 - x^2} = \rho_{sc}(x)$ the semi-circle

To do: derive $n_*(x)$ from saddle point analysis of

$\boxed{\mathcal{F}_0[n(x)] - \kappa\left(\int dx\, n(x) - 1\right)}$: energy + Lagrange multiplier today (+ κ^0)

reminder in 1D: $I = \int dx\, e^{-Nf(x)}$ gives leading contribution at minimum x_0
for $N \gg 1$

$f'(x_0) = 0 \Rightarrow$ Taylor $f(x) = f(x_0) + \frac{1}{2}(x-x_0)^2 \underbrace{f''(x_0)}_{>0}$

$$\approx \sqrt{\frac{2\pi}{N f''(x_0)}}\, e^{-Nf(x_0)}$$

here: $\frac{\delta \mathcal{F}_0[n(x)]}{\delta n(y)} \overset{!}{=} 0, \quad \frac{\partial \mathcal{F}_0[n(x)]}{\partial \kappa} = 0$

- the solution to be found must have compact support!

Literature: to Coulomb gas picture:
- S. Serfaty 1709.04089
- Peter J. Forrester: Log-Gases and RM (e.g. chapter 14.1) 2010

to general RMT
- M.L. Mehta Random Matrices, 3rd Ed. 2004
- G. Anderson, A. Guionnet, O. Zeitouni
  An Intro to RM, 2010
- Y Fyodorov Intro to RMT: GUE and beyond math-ph/0412017

Notice: $\rho_N(x) = \langle n_N(x) \rangle \rightarrow \rho_{sc}(x)$ too, to be shown

$= \int dx_2 \ldots dx_N\, \rho_N(x_1, x_2, \ldots, x_N)$

5. Saddle-point analysis of the Coulomb gas partition function

- also called steepest descent method or stationary phase approximation (c.i) of an integral with a large parameter N

$$I = \int dx\, e^{-Nf(x)}$$ expand around (minimum) extremum x_0 with $f'(x_0) = 0$

$$\Rightarrow I \approx e^{-Nf(x_0)} \underbrace{\int_{-\infty}^{\infty} dx\, e^{-N\frac{f''(x_0)}{2}(x-x_0)^2}}_{\text{"fluctuations"}} = \sqrt{\frac{2\pi}{Nf''(x_0)}}\, e^{-Nf(x_0)}$$ ($f''(x_0) > 0$)

see e.g. Fyodorov math-ph/0412017 ch 5.2.

- for several minima we have to sum over all of them
- for $\int_J$ with $J \subseteq \mathbb{R}$ and min $x_0 \notin J$ we have to expand to linear order at ∂J next to x_0

5.1 Lagrange multiplier and saddle-point equation

in $$Z_{N,\beta} = C_{N,\beta} \int \mathcal{D}[n(x)]\, \exp\left[-\beta N^2 \mathcal{F}_0[n(x)] + \left(\tfrac{\beta}{2}-1\right) N \mathcal{F}_1[n(x)] + \tfrac{\beta}{2} N(\ln N - \ln c) + o(N)\right]$$

absorb in normalisation

we have to ensure that the smoothed random measure $n(x)$ is normalised:

insert $$\delta\left[\int_{\mathbb{R}} dx\, n(x) - 1\right] = \int_{\mathbb{R}} \frac{dk}{2\pi}\, e^{ik\left(\int_{\mathbb{R}} dx\, n(x) - 1\right)}$$ using standard δ-rep.

- rescaling $ik \to \beta N^2 \kappa$ and keeping only the leading order $O(N^2)$ $\mathcal{F}_0$-contrib.

$$\Rightarrow Z_{N,\beta} \approx \tilde{C}_{N,\beta} \int \mathcal{D}[n(x)] \int_{\mathbb{R}} d\kappa\, \exp\left\{-\beta N^2 \underbrace{\left[\mathcal{F}_0[n(x)] - \kappa\left(\int dx\, n(x) - 1\right)\right]}_{\equiv S[n(x),\kappa]} + O(N)\right\}$$

Lagrange-multiplier

saddle-point $\approx$ const $e^{-\beta N^2 S[n_*(x), \kappa_*]}$

(we don't consider the fluctuations)

with $n_*(x), \kappa_*$ solving the saddle point equations (SP eq.)

$$\boxed{\begin{aligned} 0 &= \frac{\delta}{\delta n(x)} S[n(x),\kappa]\Big|_{\substack{n=n_*\\ \kappa=\kappa_*}} = \frac{x^2}{2} - \int_{\mathbb{R}} dx'\, n_*(x') \ln|x-x'| - \kappa_* \\ 0 &= \frac{\partial}{\partial \kappa} S[n(x),\kappa]\Big|_{\substack{n=n_*\\ \kappa=\kappa_*}} = \int_{\mathbb{R}} dx\, n_*(x) - 1 \end{aligned}}$$

, used $\frac{\delta n(x)}{\delta n(y)} = \delta(x-y)$

(5.9)

* the normalized solution $n_*(x)$ of the SP eq. (equilibrium measure) cannot be supported on all of $\mathbb{R}$:

for $x \gg 1$ we have $0 = \frac{x^2}{2} - \int_{\mathbb{R}} dx'\, n_*(x') \ln|x-x'| - \mu_*$

$$\approx \frac{x^2}{2} - \ln|x| \int_{\mathbb{R}} dx'\, n_*(x') = \frac{x^2}{2} - \ln|x|$$ which is inconsistent

$\Rightarrow$ the solution $n_*(x) = n_*(x; a, b)$ has compact support on $[a,b] \subseteq \mathbb{R}$

– the parameters a, b have to be determined s.th. $n_*(x; a, b)$ min F !

5.2 Derivative of the SP eq and Tricomi's solution

$\ln|x-x'|$ is not differentiable at $x = x'$

Def weak derivative: For $u(x), v(x) \in \mathcal{L}^1([a,b])$ v is the weak derivative of u if $\forall\, \varphi(x) \in \mathcal{C}^\infty$, $\varphi(a) = \varphi(b) = 0$: $\int_a^b dx\, u(x)\, \varphi'(x) = -\int_a^b dx\, v(x)\, \varphi(x)$

Here set $u(x) = \int dx'\, n_*(x') \ln|x-x'|$, we will regularise the singularity at $x = x'$ by ε

$$\int dx\, \varphi'(x) \int dx'\, n_*(x') \frac{\ln(x-x')^2}{2} = \frac{1}{2} \lim_{\varepsilon \to 0} \int dx\, \varphi'(x) \int dx'\, n_*(x') \ln[(x-x')^2 + \varepsilon^2]$$

$$= -\frac{1}{2} \int dx\, \varphi(x) \int dx'\, n_*(x') \frac{2(x-x')}{(x-x')^2 + \varepsilon^2} = -\int dx\, \varphi(x)\, \underbrace{\mathrm{Pr} \int dx'}\, \frac{n_*(x')}{x-x'}$$

or $⨍ dx'$ or $\mathrm{P.V.} \int dx'$ is Cauchy's principal value

i.e. for x singular point of $F(x)$

Def $\mathrm{Pr} \int dx'\, F(x') = \lim_{\varepsilon \to 0} \int^{x-\varepsilon} dx'\, F(x') + \int_{x+\varepsilon} dx'\, F(x')$

$\Rightarrow$ weak derivative of SP eq: $$\boxed{0 = x - \mathrm{Pr} \int dx' \frac{n_*(x')}{x-x'}} \qquad (5.15)$$

Assume that $[a,b]$ is the single interval support of $f(x)$ (why: , not true for )

Thm [Tricomi 85]

$$g(x) = \mathrm{Pr} \int_a^b dx' \frac{f(x')}{x-x'} \Rightarrow f(x) = \frac{1}{\pi \sqrt{(x-a)(b-x)}} \left(C - \mathrm{Pr} \int_a^b \frac{dt}{\pi} \frac{\sqrt{(t-a)(b-t)}}{x-t}\, g(t) \right)$$

arb. (integration) constant

here: $g(t) = t$ $\Rightarrow$ (Exercise 4.1), using $\int_a^b dx\, n_*(x) = 1$ ☺

$$n_*(x) = \frac{1}{\pi\sqrt{(x-a)(b-x)}} \cdot \left[1 - x^2 + (a+b)x + \frac{1}{8}(b-a)^2\right] \qquad (5.17)$$

* this is a solution of (5.15) for any choice of a and b!

$\Rightarrow$ we still have to determine these s.th. F is minimised.

5.3 Evaluation of $\mathcal{F}_0[u_*(x)]$ and determination of a, b

- $\int_a^b dx\, n_*(x) \cdot \underset{\text{SP eqs.}}{(5.9)} \overset{☺}{\Rightarrow} \frac{1}{2}\int_a^b dx\, n_*(x) x^2 - u_* = \int_{[a,b]^2} dx\, dx'\, n_*(x)\, n_*(x') \ln|x-x'|$

- set $x = a$ in (5.9) $\Rightarrow$ $0 = \frac{a^2}{2} - \int_a^b dx' n_*(x') \ln|a-x'| - u_*$ $\Rightarrow$ $u_* = \ldots$

$$\Rightarrow \mathcal{F}_0[u_*(x)] = \frac{1}{2}\int_a^b dx\, n_*(x) x^2 - \frac{1}{2}\int_{[a,b]^2} dx\, dx'\, n_*(x) n_*(x') \ln|x-x'|$$

$$= \frac{1}{4}\int_a^b dx\, n_*(x) x^2 + \frac{a^2}{4} - \frac{1}{2}\int_a^b dx\, n_*(x) \ln(x-a)$$

- inserting $n_*(x)$ this can be solved as a function $f(a,b)$ and minimised w.r.t. a and b. Result $a = -b = \sqrt{2}$

shortcut: we expect $n_*(x)$ to be symmetric $\Rightarrow$ $b = -a$

$$\Rightarrow n_*(x) = \frac{1}{\pi\sqrt{a^2 - x^2}}\left[1 - x^2 + \frac{1}{2}a^2\right] \quad ; \text{ for } a = \sqrt{2}$$

we get $= \frac{1}{\pi}\sqrt{2-x^2}$, so we only need to show that

$$\Pr\int_{-\sqrt{2}}^{+\sqrt{2}} dx' \frac{\sqrt{2-x'^2}}{\pi(x-x')} = \lim_{\varepsilon\to 0}\left(\int_{-\sqrt{2}}^{\varepsilon} + \int_{\varepsilon}^{\sqrt{2}}\right) dx' \frac{\sqrt{2-x'^2}}{\pi(x-x')} = X$$

-> Exercise 4.1: verify this with the antiderivative of the integrand given in eqs (5.26), (5.27) in the book LNV.

- there are other proofs for the semi-circle, e.g. using combinatorics (e.g. book A&Z):
Ex 4.3 shows that g_{sc} is the generating function for Catalan numbers.

* Q: Is the semi-circular density universal?

- for the deformation 1. = Wigner ensemble (Yes), as long as the H_{ij} are iid, that is they have the same variance
- ρ_{sc} also appears in other applications e.g. as the limiting density of the adjacency matrix of certain random graphs (Erdős-Renyi)
- for the deformation 2. = invariant ensembles (No):

for $P(H) \sim \exp(-N\beta \operatorname{Tr} \vartheta(H))$ e.g. $\vartheta(H) = \frac{1}{2}H^2 + g H^4 + \dots$

we have $\rho(x_1, \dots, x_N) = \frac{1}{Z} \prod_{k<j} |x_k - x_j|^\beta e^{-N\beta \sum_{i=1}^N \vartheta(x_i)}$ polynomial

$\Rightarrow$ in $\mathcal{F}_0[n(x)]$ replace $\int dx \frac{1}{2}x^2 n(x) \to \int dx\, \vartheta(x)\, n(x)$

$\Rightarrow \dots \Rightarrow$ SP equation $\boxed{\vartheta'(x) = \Pr \int dy \frac{n_*(y)}{x-y}}$

has a different $n_*(x) = \frac{1}{\pi}\left(\frac{1}{2} + g a^2 + 2 g x\right)\sqrt{a^2 - x^2}$, with $1 = \frac{a^2}{4} + \frac{3}{4} g a^4$ eq. for endpoints.

$\rightarrow$ Exercise 4.2 for $\vartheta(x) = x - \alpha \ln x$ gives $\rho_{MP}(x)$ Marchenko-Pastur (also called quarter-circle)

Nevertheless also for deformation 2. the local statistics e.g. the spacing distribution remain universal!

6. Derivation of the JPDF (part I) and the volume of O(N)

6.1 As a reminder: change of variables in 2 dimensions:

Consider $I_1 = \int_{\mathbb{R}^2} dx\,dy\, \rho_1(x,y)$, $I_2 = \int_{\mathbb{R}^2} dx\,dy\, \rho_2(x,y)$

with $\rho_1(x,y) = f(x^2+y^2)$, $\rho_2(x,y) = x f(x^2+y^2)$ for some $f(z)$ s.th. both integrals converge, e.g. $f(z) = e^{-z}$.

Standard strategy: change of variables from Cartesian $(x,y) \in \mathbb{R}^2$ to polar coordinates $(r,\theta) \in \mathbb{R}_+ \times [0,2\pi]$: $x = r\cos\theta$, $y = r\sin\theta$

Jacobian $$\mathcal{J}(r,\theta) = \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| = \left|\begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{pmatrix}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & +r\cos\theta \end{vmatrix} = r\underbrace{(\cos^2\theta + \sin^2\theta)}_{=1}$$

$$\Leftrightarrow \int_{\mathbb{R}^2} dx\,dy = \int_0^\infty dr \int_0^{2\pi} d\theta\, |\mathcal{J}(r,\theta)| = \int_0^\infty dr\, r \int_0^{2\pi} d\theta$$

- due to $x^2+y^2 = r^2(\cos^2\theta + \sin^2\theta) = r^2$ the integral I_1 is simple, as its integrand only depends on r:

$$I_1 = \int_0^\infty dr\, r \underbrace{\int_0^{2\pi} d\theta\, f(r^2)}_{\text{marginal } g_1(r) = r f(r^2) 2\pi} = 2\pi \int_0^\infty dr\, r\, f(r^2)$$

, while $g_2(r,\theta) = r\cos\theta\, f(r^2)$ still contains 2 integrals (here: decouple, gives 0)

6.2. The jpdf of the GOE (part I)

GOE: we have $$P(H) = C_N\, e^{-\frac{1}{2}\operatorname{Tr} H^2} = \frac{1}{2^{\frac{N}{2}} \pi^{\frac{N(N+1)}{4}}} \prod_i^N e^{-\frac{1}{2}H_{ii}^2} \prod_{i<j} e^{-H_{ij}^2}$$

s.th. $$\int_{\mathbb{R}^{N\times N}} dH\, P(H) = 1$$

- all matrices $H = H^T$, $H \in \mathbb{R}^{N\times N}$ can be diagonalised by an orthogonal trafo $H = OXO^T$, $O \in O(N)$ eigenvectors, $X = \operatorname{diag}(x_1, \dots, x_N)$

and $$\operatorname{Tr} H^2 = \operatorname{Tr}(OX\underbrace{O^T O}_{\mathbb{1}_N} XO^T) = \operatorname{Tr} X^2 = \sum_{i=1}^N x_i^2$$

* We are in the same situation as for integral I_1, we only need the Jacobian $$\mathcal{J}(X,O) = \det\left(\frac{\partial H_{ij}}{\partial(x_i, O_{ij})}\right)$$

$$dP(H) = P(H_{11}, H_{12}, \dots, H_{NN}) \underbrace{\prod_{i\leq j} dH_{ij}}_{\text{product of flat Lebesgue on } \mathbb{R}} = C_N\, e^{-\sum_{i=1}^N x_i^2} |\mathcal{J}(X,O)|\, dO \underbrace{\prod_{i=1}^N dx_i \dots dx_N}_{\text{flat } \mathcal{L}. \text{ on } \mathbb{R}^N}$$