- In any change of variables the number of variables before and after should agree: $H = H^T$ has $N$ diag $+ \frac{N(N-1)}{2}$ (strictly) upper triang. degrees of freedom (dof) $= \frac{N(N+1)}{2}$

  - $X$ has $N$ dof, $O$ has $N^2$ dof. minus the number of constraints from $OO^T = 1\!\!1_N = O^T O$: $N$ diag and $\frac{N(N-1)}{2}$ off diag
  $\Rightarrow N^2 - N - \frac{N(N-1)}{2} = \frac{N^2 - N}{2}$ ok.

- note that $O$ contains the orthonormalised eigenvectors of $H$

- $dO$ is the Haar measure on group of orthogonal matrices, it is invariant under orthogonal trafos $O' = O_1 O O_1^T$ with $O_1 O_1^T = 1\!\!1_N$ so $dO = dO'$.

$\Rightarrow$ we need to compute the integral $\int dO$ and the Jacobian

$$\boxed{J(\underline{X}, O) \overset{!}{\propto} \prod_{j>k}^{N} (x_j - x_k)}$$ to derive $g(x_1, \dots, x_N)$ incl. normalisation ← this is independent of $O$!

$\Rightarrow$ we only need $\mathrm{Vol}(O(N)) = \int dO = \frac{2^N \pi^{N^2/2}}{\Gamma_N(\frac{N}{2})}$ → Exercise for $N=2$ and general $N$

with $\Gamma_m(a) = \pi^{\frac{m(m-1)}{4}} \prod_{i=1}^{m} \Gamma\left(a - \frac{(i-1)}{2}\right)$

Theorem [Edelman] Let $H = H^T$ be a real sym. matrix with jpdf of matrix elements $P(H) = \phi(\mathrm{Tr}H, \mathrm{Tr}H^2, \dots, \mathrm{Tr}H^N)$ invariant under conjugation by an orthogonal trafo. Then the jpdf of the ordered eigenvalues $x_1 \geq x_2 \geq \dots x_N$ of $H$ reads

$$g_{ord}(x_1, \dots, x_N) = \frac{\pi^{\frac{N^2}{2}}}{\Gamma_N(\frac{N}{2})} \underbrace{\overbrace{\phi\left(\sum_i x_i, \sum_i x_i^2, \dots, \sum_i x_i^N\right)}^{\frac{1}{2^{N/2}\pi^{\frac{N(N+1)}{4}}} e^{-\frac{1}{2}\sum_{i=1}^N x_i^2}}} \prod_{j<i} (x_i - x_j).$$

→ In our example $P(H) \propto e^{-\mathrm{Tr}V(H)} \Rightarrow \exp\left[-\sum_{i=1}^{N} V(x_i)\right]$

* How are jpdf of ordered and unordered eigenvalues related

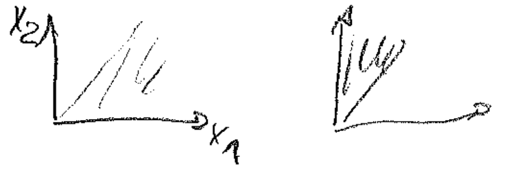
Example $N=2$:

unordered $\quad \int_{\mathbb{R}^2} dx_1 dx_2 \, \underline{e^{-V(x_1)-V(x_2)} \, |x_1-x_2|}$

invariant under permutations $x_1 \leftrightarrow x_2$

$$= \underset{x_1>x_2}{\int dx_1 \int dx_2 \, g_2(x_1,x_2)} + \underset{x_1<x_2}{\int dx_1 \int dx_2 \, g_2(x_1,x_2)} = 2 \int_{-\infty}^{\infty} dx_1 \int_{-\infty}^{x_1} dx_2 \, g_2(x_1,x_2)$$

& relabel $x_1 = x_2$, $x_2 = x_1$



Similarly one can show for general $N$ that for $g_N(x_1,\dots,x_N)$ invariant under any permutation of indices

$$\underset{\mathbb{R}^N \text{ unordered}}{\int dx_1 \dots dx_N \, g_N(x_1,\dots,x_N)} = N! \underset{x_1 \geq x_2 \geq \dots \geq x_N}{\int dx_1 \dots dx_N \, g_N(x_1,\dots,x_N)}$$

ordered, here $\prod_{i>j} |x_i - x_j| = \prod_{i>j} (x_i - x_j)$

$\Rightarrow$ taking all together for the GOE we have

$$P(H) = \prod_{i=1}^{N} \frac{e^{-H_{ii}^2/2}}{\sqrt{2\pi}} \prod_{i>j} \frac{e^{-H_{ij}^2}}{\sqrt{\pi}} = \frac{1}{(2\pi)^{\frac{N}{2}} \pi^{\frac{N^2-N}{4}}} e^{-\frac{1}{2}\sum_{i=1}^{N} x_i^2} \Rightarrow C_N e^{-\frac{1}{2}\operatorname{Tr} H^2}$$

and $g(x_1,\dots,x_N) = \int dO \, |J(X,O)| \, C_N \, e^{-\sum_{i=1}^N x_i^2/2}$

$$\overset{\text{Edelman}}{=} \underline{\int dO \frac{\pi^{\frac{N^2}{2}}}{\Gamma_N(\frac{N}{2})} \prod_{i>j} (x_i - x_j) \, e^{-\sum_{i=1}^N x_i^2/2}}$$

ordered $x_1 \geq x_2 \geq \dots \geq x_N$

$$= N! \int dO \frac{\pi^{\frac{N^2}{2}}}{\Gamma_N(\frac{N}{2})} \prod_{i>j}^{N} |x_i - x_j| \, e^{-\sum_{i=1}^N x_i^2/2} \quad \text{unordered}$$

* note that when diagonalising $H = O X O^T$ the choice of eigenvectors is not unique (with $H\vec{v}_j = x_j \vec{v}_j$ also $-\vec{v}_j$ is an eigenvector)

$\Rightarrow$ in $\int dO$ we integrate over all eigenvectors modulo this change

$\Rightarrow$ we have to divide by $2^N$: $\frac{\int dO}{2^N} = \frac{2^N \pi^{N^2/2}}{\Gamma(\frac{N}{2}) \cdot 2^N}$

$\Rightarrow$ the desired normalisation $Z_{N,\beta} = 1^{?} (2\pi)^{N/2} \prod_{j=1}^{N} \frac{\Gamma(1+j\beta/2)}{\Gamma(\beta/2)}$ from p3 ✓

# 7. The Vandermonde Determinant and Derivation of the JPDF Part (c)

- the Jacobian $J(\underline{x}, 0) \equiv \Delta_N(\{x\}) = \prod_{j>i}^{N} (x_j - x_i)$

Can be written as a determinant $= \det[x_j^{i-1}]_{i,j=1}^{N} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & & x_N \\ \vdots & \vdots & & \vdots \\ x_1^{N-1} & x_2^{N-1} & \cdots & x_N^{N-1} \end{vmatrix}$

Proof: Exercise ($N=2$ trivial)

## 7.1. Properties of the Vandermonde-determinant:

1) $\Delta_N(\{x\})$ is a totally antisymmetric polynomial in all $x_1, \dots, x_N$ as any exchange of 2 columns in the det gives a minus

2) $\Delta_N(\{x\})$ contains $\frac{N(N-1)}{2}$ factors (which is easy to see by induction)

$\left[\Delta_N(\{x\}) = \prod_{i=1}^{N-1} (x_N - x_i)\, \underbrace{\Delta_{N-1}(\{x\})}\right]$ only contains $x_1, \dots, x_{N-1}$

$\Rightarrow$ rescaling all $x_i$ by the same constant $x_i \to a x_i$ (e.g. $a = \sqrt{\beta N}$)

yields $\Delta_N(\{ax\}) = a^{\frac{N(N-1)}{2}} \Delta_N(\{x\})$

3) Linear Algebra properties of $\Delta_N$:

multiples of rows can be added to each other without changing the value of the det:

$$\Rightarrow \Delta_N(\{x\}) = \begin{vmatrix} 1 & & 1 \\ x_1 & & x_N \\ \vdots & & \\ \hat{\pi}_{N-1}(x_1) & & \hat{\pi}_{N-1}(x_N) \end{vmatrix} = \dots = \begin{vmatrix} \hat{\pi}_0(x_1) & \cdots & \hat{\pi}_0(x_N) \\ \hat{\pi}_1(x_1) & & \hat{\pi}_1(x_N) \\ \vdots & & \vdots \\ \hat{\pi}_{N-1}(x_1) & \cdots & \hat{\pi}_{N-1}(x_N) \end{vmatrix} = \frac{1}{a_0^{(0)} a_1^{(1)} \cdots a_{N-1}^{(N-1)}} \begin{vmatrix} \pi_0(x_1) & \cdots & \pi_0(x_N) \\ & & \\ \pi_{N-1}(x_1) & \cdots & \pi_{N-1}(x_N) \end{vmatrix}$$

by adding mult.

$\hat{\pi}_{N-1}(x) = x^{N-1} + a_{N-2}^{(N-1)} x^{N-2} + \dots + a_1^{(N-1)} x + a_0^{(N-1)}$; can obtain arbitrary polyn $\pi_{N-1}(x) = a_{N-1}^{(N-1)} x^{N-1} + \dots$

arbitrary monic polynomial

by mult in and out, provided all leading coeff $\neq 0$

Example: for the (probabilist's) Hermite polynomials $He_n(x)$

orthogonal w.r.t. $e^{-\frac{x^2}{2}}$: $\int_{-\infty}^{\infty} dx\, e^{-\frac{x^2}{2}} He_n(x) He_m(x) \sim \delta_{nm}$

we have $$\Delta_N(\{x\}) = \begin{vmatrix} He_0(x_1) & \dots & He_0(x_N) \\ \vdots & & \\ He_{N-1}(x_1) & \dots & He_{N-1}(x_N) \end{vmatrix}$$ as $He_n(x) = x^n + O(x^{n-1})$ are monic

remark: the Hermite polynomials $H_n(x)$ orth. w.r.t. $e^{-x^2}$, $\int_{-\infty}^{\infty} dx\, e^{-x^2} H_n(x) H_m(x) \sim \delta_{nm}$

yield prefactors as $H_n(x) = 2^n x^n + O(x^{n-1})$ are not monic, $a_n^{(n)} = 2^n$

$$\Delta_N(\{x\}) = \frac{1}{1 \cdot 2 \cdots 2^{N-1}} \begin{vmatrix} H_0(x_1) & \dots & H_0(x_N) \\ H_{N-1}(x_1) & \dots & H_{N-1}(x_N) \end{vmatrix}$$

## 7.2 Derivation of the jpdf Part II:

Here we do not rely on the theorem by Edelman and thus on the invariance of $P(H)$ under rotations $H \to OHO^T$.

Consider an infinitesimal transformation of $\boxed{H = O\underline{X}O^T}$ or differentiation

$\Rightarrow dH = dO\,\underline{X}O^T + O\,d\underline{X}\,O^T + O\underline{X}\,dO^T$ , with $dO, d\underline{X}, dO^T$ matrices

using $\boxed{OO^T = \mathbb{1}_N = O^TO}$ we have $d(OO^T) = dO\,O^T + O\,dO^T = 0$

i.e. $\underbrace{(dO\,O^T)^T = O\,dO^T = -dO\,O^T}$ $= d(O^TO) = dO^T O + O^T dO$

is antisymmetric, as is $\boxed{d\Omega \equiv O^T dO}$

$$\Rightarrow dH = \overbrace{OO^T}^{\mathbb{1}} \underbrace{dO}_{d\Omega}\,\underline{X}\,O^T + O\,d\underline{X}\,O^T + O\underline{X}\,\underbrace{dO^T O}_{d\Omega^T = -d\Omega}\overbrace{O^T}^{\mathbb{1}}$$

$$= O(d\Omega\,\underline{X} + d\underline{X} - \underline{X}\,d\Omega)\,O^T = O\underbrace{(d\underline{X} + [d\Omega, \underline{X}])}_{d\hat{H}}O^T$$

because $dH$ and $d\hat{H}$ are related by an orthogonal trafo, it is sufficient to find the Jacobian for $d\hat{H}$: $\frac{\partial \hat{H}_{ij}}{\partial(x_i, \Omega_{ij})}$

using that $\bar X = \mathrm{diag}(x_1, \dots, x_N)$ and thus $d\bar X$ are <u>diagonal</u> we have

$$d\hat H_{ij} = d\bar X_{ij} + (d\Omega \bar X - \bar X d\Omega)_{ij} = dx_i \delta_{ij} + d\Omega_{ij} x_j - x_i d\Omega_{ij}$$

$$\Rightarrow \boxed{\frac{d\hat H_{ij}}{dx_k} = \delta_{ik}\delta_{ij} \quad , \quad \frac{d\hat H_{ij}}{d\Omega_{kl}} = \delta_{ik}\delta_{jl}(x_j - x_i)}$$

<u>Example $N = 3$:</u> $\exists\ \frac{N(N+1)}{2} = 6$ indep. d.o.f.

$d\Omega = -d\Omega^T \Rightarrow$ parametrise $d\Omega = \begin{pmatrix} 0 & d\Omega_{12} & d\Omega_{13} \\ -d\Omega_{12} & 0 & d\Omega_{23} \\ -d\Omega_{13} & -d\Omega_{23} & 0 \end{pmatrix}$ and $dx_1, dx_2, dx_3$

$$\Rightarrow \begin{pmatrix} \frac{d\hat H_{11}}{d\Omega_{12}} & \frac{d\hat H_{11}}{d\Omega_{13}} & \frac{d\hat H_{11}}{d\Omega_{23}} & \frac{d\hat H_{11}}{dx_1} & \frac{d\hat H_{11}}{dx_2} & \frac{d\hat H_{11}}{dx_3} \\ \frac{d\hat H_{12}}{d\Omega_{12}} & & & & & \\ \frac{d\hat H_{13}}{d\Omega_{12}} & & & & & \\ \frac{d\hat H_{22}}{d\Omega_{12}} & & & & & \\ \frac{d\hat H_{13}}{d\Omega_{12}} & & & & & \\ \frac{d\hat H_{23}}{d\Omega_{12}} & & & & & \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ x_2 - x_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & x_3 - x_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x_3 - x_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

reorder columns to bring to diag form

- we have to compute $\left| \det \left[ \frac{\partial \hat H_{ij}}{\partial x_k \Omega_{lm}} \right] \right| = \begin{vmatrix} 1 & & & & & \\ & x_2 - x_1 & & & & \\ & & x_3 - x_1 & & & \\ & & & 1 & & \\ & & & & x_3 - x_2 & \\ & & & & & 1 \end{vmatrix}$

$$= |(x_2 - x_1)(x_3 - x_1)(x_3 - x_2)|$$

$$= |\Delta_3(\{x\})| .$$

Also for general $N$ such an ordering with diagonal Jacobi-determinant can be found

$\boxed{\frac{dH_{11}}{dx_1}}$ $\frac{dH_{12}}{dx_1}$ $\frac{dH_{13}}{dx_1}$ $\frac{dH_{1N}}{dx_1}$ $\frac{dH_{22}}{dx_1}$ $\frac{dH_{23}}{dx_1}$ $\leftarrow$ order $H$

$\downarrow$ order

$x_1$ ⋮ $x_N$ $\begin{pmatrix} 0 & -\Omega_{12} & \cdots \\ & 0 & \\ & & \ddots \end{pmatrix}$

$\frac{dH_{11}}{d\Omega_{12}}$ $\boxed{\frac{dH_{12}}{d\Omega_{12}}}$

$\frac{dH_{11}}{d\Omega_{13}}$ $\frac{dH_{12}}{d\Omega_{13}}$ $\boxed{\frac{dH_{13}}{d\Omega_{13}}}$

⋮

$\frac{dH_{11}}{d\Omega_{1N}}$ $\boxed{\frac{dH_{1N}}{d\Omega_{1N}}}$

$\frac{dH_{11}}{dx_2}$ $\boxed{\frac{dH_{22}}{dx_2}}$ $\frac{dH_{23}}{dx_2}$

$\frac{dH_{11}}{d\Omega_{23}}$ $\boxed{\frac{dH_{23}}{d\Omega_{23}}}$

$\frac{dH_{11}}{d\Omega_{2N}}$

$\frac{dH_{11}}{dx_3}$ $\boxed{\frac{dH_{33}}{dx_3}}$

$\frac{dH_{11}}{d\Omega_{34}}$

⋮

with the result that $\left|\det\left(\frac{\partial H}{\partial x\Omega}\right)\right| = \left|\Delta_N(\{x\})\right|$