Summary

- IRTG-meeting ⇒ Exam conditions: in due course of the lec. have stopped
  ~30 min oral exam with me & assistant
  prepare chapter for a short presentation (paper/board) + Q's
- remaining students – for full 5 ECTS credit points:
  * participation in exercises, regular attendance and $\geq 2\times$ presenting solutions
  * ~30 min oral exam, just Q&A

today evaluation ≈ 11:30

summarize properties of orthonormal polynomials (approach on $\mathbb{R}$)

[Chapter 10 LNV]

$$x P_n(x) = c_{n+1} P_{n+1}(x) + a_n P_n(x) + c_n P_{n-1}(x)$$

Kernel $K_N(x,y) = \sum_{i=0}^{N-1} P_i(x) P_i(y) \sim \frac{P_N(x) P_{N-1}(y) - P_N(y) P_{N-1}(x)}{x-y}$ CD

⇒ Normalisation of jpdf = partition function $Z_N \sim N! \prod_{i=0}^{N-1} h_i$, $c_n = \sqrt{\frac{h_n}{h_{n-1}}}$

$\|\tilde{P}_n\|^2$ monic op

rep of $P_N(x) = \langle \det(x-H) \rangle_N$ monic

$$K_{N+1}(x,y) = h_N^{-1} \langle \det(x-H) \det(y-H) \rangle_N$$

Dyson-Mehta Thm: From $\int dx\, w(x) K_N(x_1,x) K_N(x,x_2) = K_N(x_1,x_2)$, $\int dx\, w(x) K_N(x,x) = c$

it follows $\int dx_N w(x_N) \det[K_N(x_i,x_j)]_{i,j=1}^{N} = (c-N+1) \det[K_N(x_i,x_j)]_{i,j=1}^{N-1}$ (reduced dim.)

Why is this useful?

$$P_{jpdf}(x_1,\dots,x_N) = \prod_{i=1}^N w(x_i)\, |\Delta_N(x)|^2 = \prod_{i=1}^N h_{i-1} w(x_i) \det[K_N(x_i,x_j)]_{i,j=1}^N$$

→ apply Thm:

- to do all remaining $(N-1)\cdot(N-1)!$ perm start again with Id (which we have counted already) and do 1 pair exchange with the second argument of the last kernel.

$$K(x_1,x_1)\ \dots\ K(x_j\, x_j)\ \dots\ K(x_N, x_N)$$

- there are $N-1$ possibilities (positions) to do so, we get a "$(-)$" from the 1 pair exchange

- $\int dx_N\, K(x_j, x_N)\, K(x_N, x_j) = K(x_j, x_j)$

giving $-$ Id of $N-1$ variables. Now we do the remaining $(N-1)!$ permutations of the $(N-1)$ arguments yielding

$$\det_{1..N-1} K$$

$$\Rightarrow \quad \int dx_N \det_{1..N}\left[K(x_i, x_j)\right] = \left(c-(N-1)\right)\det_{1..N-1}\left[K(x_i, x_j)\right]$$

$\square$

[alternatively see Mehta, Th 5.1.4 p 75, 76]

$\Rightarrow$ compute all $\beta=2$ correlation functions

$$K^{(\beta=2)}_{N,k}(x_1 \dots x_k) = \frac{\cancel{N!}}{(N-k)!}\int dx_{k+1} \dots dx_N \prod_{j=1}^{N}\omega(x_j)\, \Delta_N(\{x\})^2 \cdot \frac{1}{\cancel{N!}\prod_{j=0}^{N-1} h_j}$$

$$= \frac{\cancel{N!}}{(N-k)!}\;\underbrace{\int dx_{k+1}\dots dx_N \cancel{\prod_{j=0}^{N-1} h_j}}_{\cancel{N!\prod_{j=0}^{N-1} h_j}}\;\det\underbrace{\left[\omega(x_i)^{\frac{1}{2}}\omega(x_j)^{\frac{1}{2}}\sum_{\ell=0}^{N-1} P_\ell(x_i) P_\ell(x_j)\right]}_{K(x_i, x_j)\text{, with } c=N}$$

- the integrations yield

$$\int dx_N\ (N-(N-1))\det_N = \det_{N-1}$$

$$\int dx_{N-1} \det_{N-1} = \left(N-(N-1-1)\right)\det_{N-2}$$

$$\int dx_{k+1} \det_{k+1} = \left(N-(k+1-1)\right)\det_{k}$$

$$\Rightarrow R^{(\beta=2)}_{N,k}(x_1,\dots,x_k) = \frac{(N-k)!}{(N-k)!} \det_{1..k}\left[\omega(x_i)^{\frac{1}{2}}\omega(x_j)^{\frac{1}{2}}\sum_{l=0}^{N-1} P_l(x_i) P_l(x_j)\right]$$

$$= \prod_{l=1}^{k} \omega(x_l) \det_{1..k}\left[K_N(x_i, x_j)\right]$$

some authors define the kernel with or without the weights

You will also find the notation of a

$\underline{\text{wave-function}}$ $\underline{\Psi_k(x) = \omega(x)^{\frac{1}{2}} P_k(x)}$

as then the Ψ are orthonormal functions

$$\boxed{\int dx\, \Psi_k(x)\Psi_l(x) = \delta_{kl}}$$

$$\Rightarrow R^{(\beta=2)}_{N,k}(x_1,\dots,x_k) = \det_{1\le i,j\le k}\left[\sum_{l=0}^{N-1}\Psi_l(x_i)\Psi_l(x_j)\right]$$

b.w.

9. The GOE & GSE [chapt. 12 LNK].

- all correlation functions for $\beta = 1, 4$ can be expressed as a Pfaffian ($\mathrm{Pf}\,A = \det A^{\frac{1}{2}}$) or Qdet of a quaternion-values (2x2 matrix valued) kernel of skew-orthogonal polynomials

scalar products: $\langle f, g\rangle_2 = \int f(x)\, g(x)\, \omega(x)\, dx$ $\quad \beta = 2$ ordinarily

skew " ": $\langle f, g\rangle_4 = \int [f(x) g'(x) - f'(x) g(x)]\,\omega(x)\, dx = -\langle g, f\rangle_4$ $\quad \beta = 4$

$\langle f, g\rangle_1 = \iint f(x) g(y)\, \mathcal{E}(x-y)\, \omega(x)\omega(y)\, dx\, dy = -\langle g, f\rangle_1$ $\quad \beta = 1$

where $\mathcal{E}(x) = \begin{cases} +\frac{1}{2} & x > 0 \\ -\frac{1}{2} & x < 0 \end{cases}$

8.4 Example GUE – k-point correlation functions:

previous lecture $\tilde{P}_n(x) = 2^{-n} H_n(x) = x^n + \mathcal{O}(x^n)$ monic OP

wrt $\omega(x) = e^{-x^2}$, $H_n(x)$ Hermite polynomials (different from book LNV: probab. $He_n(x)$: $\omega = e^{-x^2/2}$)

with $\int_{-\infty}^{\infty} dx\, e^{-x^2} \tilde{P}_n(x) \tilde{P}_m(x) = \delta_{nm} \sqrt{\pi}\, 2^{-n} n! = \delta_{nm} h_n$, $P_n(x) = \frac{\tilde{P}_n(x)}{\sqrt{h_n}}$

$$\Rightarrow K_N(x,y) = \sum_{\ell=0}^{N-1} \frac{\tilde{P}_\ell(x) \tilde{P}_\ell(y)}{h_\ell} = \sum_{\ell=0}^{N-1} P_\ell(x) P_\ell(y)$$

$$= \sum_{\ell=0}^{N-1} \frac{2^{-2\ell} H_\ell(x) H_\ell(y)}{\sqrt{\pi}\, 2^{-\ell} \ell!} = c_N \frac{P_N(x) P_{N-1}(y) - P_N(y) P_{N-1}(x)}{x-y}$$

$c_N = \sqrt{\frac{h_N}{h_{N-1}}}$

$\Rightarrow$ all $R_{N,k}(x_1, \ldots, x_k)$, drastic simplification from $(N-k)$-fold integral

Spectral density $\boxed{R_{N,1}(x) = \omega(x) K_N(x,x)}$ (from $(N-1)$ integrals) recall $\int dx\, R_{N,1}(x) = N$ unnormed.

l'Hopital: $P_N(y) = P_N(x) + (y-x) P_N'(x) + \mathcal{O}(y-x)^2$

$$\Rightarrow \lim_{y \to x} K_N(x,y) = \lim_{x \to y} \frac{c_N}{(x-y)} \Big(P_N(x) (P_{N-1}(x) + (y-x) P_{N-1}'(x) + \ldots) - (P_N(x) + (y-x) P_N'(x) + \ldots) P_{N-1}(x) \Big)$$

$$= c_N \left(P_N'(x) P_{N-1}(x) - P_N(x) P_{N-1}'(x) \right)$$

Ex. 6.2: $H_n'(x) = 2n H_{n-1}(x)$

$$\Rightarrow \underline{R_{N,1}(x)} = \sqrt{\frac{h_N}{h_{N-1}}} \left(\frac{\tilde{P}_N'(x)}{\sqrt{h_N}} \frac{\tilde{P}_{N-1}(x)}{\sqrt{h_{N-1}}} - \frac{\tilde{P}_N(x)}{\sqrt{h_N}} \frac{\tilde{P}_{N-1}'(x)}{\sqrt{h_{N-1}}} \right) e^{-x^2}$$

$$= e^{-x^2} \frac{2^{-(2N-1)}}{h_{N-1}} \left(H_N'(x) H_{N-1}(x) - H_N(x) H_{N-1}'(x) \right) = \frac{e^{-x^2}}{\sqrt{\pi} (N-1)!\, 2^N} \left(2N H_{N-1}^2(x) - 2(N-1) H_N(x) H_{N-2}(x) \right)$$

plot and compare with numerical simulation.

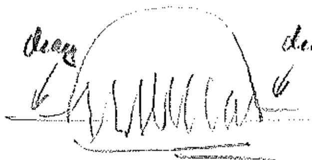
* these exact representations for finite-N are very useful for $N\to\infty$

e.g. the semi-circle can be derived from Hermite-polynomials (not completely straightforward, see chapt. 10 LNV) this is the global density

→ local density: the sine-kernel:

heuristics: the zeros of the OP become dense on the support of $\lim_{N\to\infty} \frac{P_N(x)}{N} = \rho_{sc}(x)$

decay   decay   i.e. for increasing degree the $P_N(x)$ oscillate strongly!



osc region, ∃ zeros

$H_n(x)$ $n \gg 1$

$$\boxed{\begin{aligned} &\text{even} \quad H_{2n}(x) = (-1)^n 2^n (2n-1)!!\, e^{\frac{x^2}{2}} \left[\cos(\sqrt{4n+1}\,x) + O(n^{-\frac{1}{2}})\right] \quad \text{cosine} \\ &\text{odd} \quad H_{2n+1}(x) = (-1)^n 2^{n+\frac{1}{2}} (2n-1)!!\, e^{\frac{x^2}{2}} \sqrt{2n+1} \left[\sin(\sqrt{4n+3}\,x) + O(n^{-\frac{1}{2}})\right] \quad \text{sine} \end{aligned}}$$

→ see Gradshteyn-Ryzhik 8.955, can be derived from saddle point approx of the integral representation $H_n(x) = \frac{2^n}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt\, (x+it)^n e^{-t^2}$

- we zoom into the origin, $x = \frac{\hat{x}}{\sqrt{N}}$, keeping $\hat{x}$ fixed when $N\to\infty$

$$\Rightarrow \left. w(x)^{\frac{1}{2}} w(y)^{\frac{1}{2}} K_N\!\left(\tfrac{\hat{x}}{\sqrt{N}}, \tfrac{\hat{y}}{\sqrt{N}}\right) \right) \overset{N\to\infty}{\sim} \frac{1}{2(\hat{x}-\hat{y})} \left(\cos(2\hat{x})\sin(2\hat{y}) - \sin(2\hat{x})\cos(2\hat{y})\right)$$

dropping all const   Christoffel-Darboux

$$= -\frac{\sin(2(\hat{x}-\hat{y}))}{2(\hat{x}-\hat{y})} = -K_{\sin}(\hat{x},\hat{y})$$

(or obtain from $\sum_{n=0}^{N-1} \to \int_0^1 dt$ with $t = \frac{n}{N}$: even, odd

$$\int_0^1 dt\, \underbrace{\cos(2t\hat{x})\cos(2t\hat{y}) + \sin(2t\hat{x})\sin(2t\hat{y})}_{\cos(2t(\hat{x}-\hat{y}))} = \left.\frac{\sin(2t(\hat{x}-\hat{y}))}{2(\hat{x}-\hat{y})}\right|_0^1 \checkmark)$$

more general OP wrt $e^{-V(x)}$ oscillate asymptotically in the same way

→ in the limit $N\to\infty$ lead also to $K_{\sin}$ = Universality!