Lösungen: Vorkurs Zettel 12

Aufgabe 1

a)

$$\begin{aligned}
\vec{V}(x,y,z) &= \begin{pmatrix} x \\ y \\ z \end{pmatrix} \\
\nabla \cdot \vec{V} &= \partial_x x + \partial_y y + \partial_z z = 1 + 1 + 1 \\
&= 3 \\
\vec{\nabla} \times \vec{V} &= \begin{pmatrix} \partial_y z - \partial_z y \\ \partial_z x - \partial_x z \\ \partial_x y - \partial_y x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

b)

$$\begin{aligned}
\vec{V}(x,y,z) &= \begin{pmatrix} y \\ z \\ x \end{pmatrix} \\
\nabla \cdot \vec{V} &= \partial_x y + \partial_y z + \partial_z x \\
&= 0 \\
\vec{\nabla} \times \vec{V} &= \begin{pmatrix} \partial_y x - \partial_z z \\ \partial_z y - \partial_x x \\ \partial_x z - \partial_y y \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \\ -1 \end{pmatrix}
\end{aligned}$$

c)

$$\begin{aligned}
\vec{V}(x,y,z) &= \begin{pmatrix} y \\ x \\ 0 \end{pmatrix} \\
\nabla \cdot \vec{V} &= \partial_x y + \partial_y x + \partial_z 0 \\
&= 0 \\
\vec{\nabla} \times \vec{V} &= \begin{pmatrix} \partial_y x - \partial_z 0 \\ \partial_z y - \partial_x 0 \\ \partial_x x - \partial_y y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}
\end{aligned}$$

d)

$$
\begin{aligned}
\vec{V}(x,y,z) &= \begin{pmatrix} x^2z^2 \\ y\exp[y] \\ x\log(z) \end{pmatrix} \\
\nabla\cdot\vec{V} &= \partial_x(x^2z^2) + \partial_y(y\exp[y]) + \partial_z(x\log(z)) \\
&= 2xz^2 + \exp[y] + y\exp[y] + \frac{x}{z} \\
&= x\left(2z^2 + \frac{1}{z}\right) + \exp[y](y+1) \\
\vec{\nabla}\times\vec{V} &= \begin{pmatrix} \partial_y x\log(z) - \partial_z y\exp[y] \\ \partial_z x^2z^2 - \partial_x x\log(z) \\ \partial_x y\exp[y] - \partial_y x^2z^2 \end{pmatrix} = \begin{pmatrix} 0 \\ 2zx^2 - \log(z) \\ 0 \end{pmatrix} \\
&= \left(2zx^2 - \log(z)\right)\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}
\end{aligned}
$$

Aufgabe 2

$$
\begin{aligned}
\vec{V}(\vec{x}) &= \begin{pmatrix} \alpha x + (\beta+\gamma)y \\ (\beta-\gamma)x + \alpha y \\ 0 \end{pmatrix} \\
\nabla\cdot\vec{V} &= \partial_x(\alpha x + (\beta+\gamma)y) + \partial_y((\beta-\gamma)x + \alpha y) + \partial_z 0 \\
&= 2\alpha \\
\Rightarrow \quad \nabla\cdot\vec{V} = 0 &\quad \Leftrightarrow \quad \alpha = 0
\end{aligned}
$$

$$
\begin{aligned}
\nabla\times\vec{V} &= \begin{pmatrix} \partial_y 0 - \partial_z((\beta-\gamma)x + \alpha y) \\ \partial_z(\alpha x + (\beta+\gamma)y) - \partial_x 0 \\ \partial_x((\beta-\gamma)x + \alpha y) - \partial_y(\alpha x + (\beta+\gamma)y) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ (\beta-\gamma) - (\beta+\gamma) \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 0 \\ -2\gamma \end{pmatrix} \\
\Rightarrow \quad \nabla\times\vec{V} = \vec{0} &\quad \Leftrightarrow \quad \gamma = 0
\end{aligned}
$$

Aufgabe 3

$$
\begin{aligned}
\nabla\cdot(\nabla\times\vec{A}) &= \nabla\cdot\begin{pmatrix} \partial_y A_z - \partial_z A_y \\ \partial_z A_x - \partial_x A_z \\ \partial_x A_y - \partial_y A_x \end{pmatrix} \\
&= \partial_x(\partial_y A_z - \partial_z A_y) + \partial_y(\partial_z A_x - \partial_x A_Z) + \partial_z(\partial_x A_y - \partial_y A_x) \\
&= \partial_x\partial_y A_z - \partial_x\partial_z A_y + \partial_y\partial_z A_x - \partial_y\partial_x A_z + \partial_z\partial_x A_y - \partial_z\partial_y A_x \\
&= 0
\end{aligned}
$$

Recherche

Der Quabla-Operator ist der sog. d'Alembert-Operator $\Box$ und hat die Form

$$\Box = \partial_t^2 - \sum_{i=1}^{d} \partial_{x_i}^2 = \partial_t^2 - \nabla^2$$