Lösungen: Vorkurs Zettel 4

Wirtschaftsübung

$$\begin{aligned} & x_1 = 0.8 \cdot x \quad \Rightarrow \quad x_2 = 1.1 \cdot x_1 = 1.1 \cdot 0.8 \cdot x \\ \Rightarrow \quad & x_3 = 1.1 \cdot x_2 = 1.1 \cdot 1.1 \cdot 0.8 \cdot x \\ & \quad = 0.968 \cdot x \end{aligned}$$

Aufgabe 1

a)

$$\begin{aligned} \sum_{n=3}^{6} (n-1)^2 &= \sum_{n=2}^{5} n^2 \\ &= 2^2 + 3^2 + 4^2 + 5^2 = 4 + 9 + 16 + 25 \\ &= 54 \end{aligned}$$

b)

$$\begin{aligned} \sum_{n=-1}^{4} n(n+1) &= \sum_{n=1}^{4} n(n+1) \\ &= 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 = 2 + 6 + 12 + 20 \\ &= 40 \end{aligned}$$

c)

$$\begin{aligned} \sum_{n=0}^{4} n! &= 0! + 1! + 2! + 3! + 4! \\ &= 1 + 1 + 2 + 6 + 24 \\ &= 34 \end{aligned}$$

d)

$$\begin{aligned} \sum_{n=0}^{5} (n + x^n) &= 0 + x^0 + 1 + x^1 + 2 + x^2 + 3 + x^3 + 4 + x^4 + 5 + x^5 \\ &= 1 + 1 + 2 + 3 + 4 + 5 + x + x^2 + x^3 + x^4 + x^5 \\ &= 16 + x + x^2 + x^3 + x^4 + x^5 \end{aligned}$$

Aufgabe 2

a)

$$\frac{1}{3}+\frac{2}{5}+\frac{4}{7}+\frac{8}{9}+\ldots=\frac{2^0}{2\cdot 0+3}+\frac{2^1}{2\cdot 1+3}+\frac{2^2}{2\cdot 2+3}+\frac{2^3}{2\cdot 3+3}+\ldots$$
$$=\sum_{n=0}^{\infty}\frac{2^n}{2n+3}$$

b)

$$\frac{1}{4}-\frac{1}{8}+\frac{1}{16}-\frac{1}{32}+\ldots=\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}-\frac{1}{2^5}+\ldots$$
$$=\frac{(-1)^0}{2^{2+0}}+\frac{(-1)^1}{2^{2+1}}+\frac{(-1)^2}{2^{2+2}}+\frac{(-1)^3}{2^{2+3}}+\ldots$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{2+n}}$$

Aufgabe 3

a)

$$f(x)=\sin(x)=\sum_{k=0}^{\infty}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\sin(x)\bigg|_{x=x_0}\frac{(x-x_0)^k}{k!}$$
$$\overset{x_0=0}{=}\sin(0)+\frac{\mathrm{d}}{\mathrm{d}x}\sin(x)\bigg|_{x=0}x+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\sin(x)\bigg|_{x=0}\frac{x^2}{2}$$
$$+\frac{\mathrm{d}^3}{\mathrm{d}x^3}\sin(x)\bigg|_{x=0}\frac{x^3}{6}+\frac{\mathrm{d}^4}{\mathrm{d}x^4}\sin(x)\bigg|_{x=0}\frac{x^4}{24}+\frac{\mathrm{d}^5}{\mathrm{d}x^5}\sin(x)\bigg|_{x=0}\frac{x^5}{120}+\mathcal{O}\left(x^6\right)$$
$$=\cos(0)x-\sin(0)\frac{x^2}{2}-\cos(0)\frac{x^3}{6}+\sin(0)\frac{x^4}{24}+\cos(0)\frac{x^5}{120}+\mathcal{O}\left(x^6\right)$$
$$=x-\frac{x^3}{6}+\frac{x^5}{120}+\mathcal{O}\left(x^6\right)$$

b)

$$f(x)=\cos(x)=\sum_{k=0}^{\infty}\frac{\mathrm{d}^k}{\mathrm{d}x^k}\cos(x)\bigg|_{x=x_0}\frac{(x-x_0)^k}{k!}$$
$$\overset{x_0=0}{=}\cos(0)+\frac{\mathrm{d}}{\mathrm{d}x}\cos(x)\bigg|_{x=0}x+\frac{\mathrm{d}^2}{\mathrm{d}x^2}\cos(x)\bigg|_{x=0}\frac{x^2}{2}$$
$$+\frac{\mathrm{d}^3}{\mathrm{d}x^3}\cos(x)\bigg|_{x=0}\frac{x^3}{6}+\frac{\mathrm{d}^4}{\mathrm{d}x^4}\cos(x)\bigg|_{x=0}\frac{x^4}{24}+\frac{\mathrm{d}^5}{\mathrm{d}x^5}\cos(x)\bigg|_{x=0}\frac{x^5}{120}+\mathcal{O}\left(x^6\right)$$
$$=\cos(0)-\sin(0)x-\cos(0)\frac{x^2}{2}+\sin(0)\frac{x^3}{6}+\cos(0)\frac{x^4}{24}-\sin(0)\frac{x^5}{120}+\mathcal{O}\left(x^6\right)$$
$$=1-\frac{x^2}{2}+\frac{x^4}{24}+\mathcal{O}\left(x^6\right)$$

c)

Behauptung:
Für jedes Polynom $p_n(x) = \sum_{k=0}^{n} a_k x^k$, mit $a_0, \dots, a_n \in \mathbb{R}$, $n \in \mathbb{N}$ gilt:

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k} p_n(x) \bigg|_{x=0} = \begin{cases} k! a_k, & k \leqslant n, \\ 0, & k > n. \end{cases}$$

Proof. Seien $n \in \mathbb{N}$ und $k \leqslant n$. Dann gilt:

$$\begin{aligned} \frac{\mathrm{d}^k}{\mathrm{d}x^k} p_n(x) = \frac{\mathrm{d}^k}{\mathrm{d}x^k} \sum_{s=0}^{n} a_s x^s = \sum_{s=0}^{n} a_s \frac{\mathrm{d}^k}{\mathrm{d}x^k} x^s \\ = \sum_{s=k}^{n} a_s \frac{s!}{(s-k)!} x^{s-k}. \end{aligned}$$

Betrachte $x = 0$:

$$\begin{aligned} \frac{\mathrm{d}^k}{\mathrm{d}x^k} p_n(x) \bigg|_{x=0} &= \sum_{s=k}^{n} a_s \frac{s!}{(s-k)!} x^{s-k} \bigg|_{x=0} \\ &= k! a_k. \end{aligned}$$

Sei nun $k > n$. Dann gilt für alle $s \leqslant n$:

$$\begin{aligned} \frac{\mathrm{d}^k}{\mathrm{d}x^k} a_s x^s &= \frac{\mathrm{d}^k - s}{\mathrm{d}x^k - s} s! a_s \\ &= 0 \end{aligned}$$

□

Mit der obigen Behauptung folgt für jedes Polynom:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{\mathrm{d}^k}{\mathrm{d}x^k} p_n(x) \bigg|_{x=0} \frac{x^k}{k!} &= \sum_{k=0}^{n} k! a_k \frac{x^k}{k!} \\ &= \sum_{k=0}^{n} a_k x^k = p_n(x) \end{aligned}$$