Lösungen: Vorkurs Zettel 8

Aufgabe 1

a)

$$\begin{aligned}\int_2^4 \mathrm{d}x\,(x^2+4x+42) &= \frac{x^3}{3}+2x^2+42x\bigg|_2^4\\ &= \frac{64}{3}+2\cdot 16+42\cdot 4-\frac{8}{3}-2\cdot 4-42\cdot 2\\ &= \frac{380}{3}\end{aligned}$$

b)

$$\begin{aligned}\int_2^4 \mathrm{d}y\,(x^2+4y+42z) &= x^2y+2y^2+42yz\bigg|_2^4\\ &= 4x^2+2\cdot 16+42\cdot 4z-2x^2-2\cdot 4-42\cdot 2z\\ &= 2x^2+84z+24\end{aligned}$$

c)

$$\begin{aligned}\int_{-\frac{1}{\alpha}}^0 \mathrm{d}x\,\exp\left[\alpha x\right] &= \frac{1}{\alpha}\exp\left[\alpha x\right]\bigg|_{-\frac{1}{\alpha}}^0\\ &= \frac{1}{\alpha}\left(1-\frac{1}{e}\right)\end{aligned}$$

d)

$$\begin{aligned}\int_1^5 \mathrm{d}x\,\frac{1}{x} &= \log(x)\bigg|_1^5\\ &= \log(5)-\log(1)\\ &= \log(5)\end{aligned}$$

Aufgaben partielle Integration

a)

$$\begin{aligned}\int_0^\pi \mathrm{d}x\, x\cos(x) &= x\sin(x)\Big|_0^\pi - \int_0^\pi \mathrm{d}x\ \sin(x)\\ &= x\sin(x) + \cos(x)\Big|_0^\pi\\ &= \cos(\pi) - \cos(0)\\ &= -2\end{aligned}$$

b)

$$\begin{aligned}\int_0^e \mathrm{d}x\, x\log(x) &= \frac{x^2}{2}\log(x)\Big|_1^e - \int_1^e \mathrm{d}x\ \frac{x^2}{2}\frac{1}{x}\\ &= \frac{x^2}{2}\log(x)\Big|_1^e - \int_1^e \mathrm{d}x\ \frac{x}{2}\\ &= \frac{x^2}{2}\left(\log(x) - \frac{1}{2}\right)\Big|_1^e\\ &= \frac{1}{4}\left(e^2+1\right)\end{aligned}$$

c)

$$\begin{aligned}\int_0^4 \mathrm{d}x\,(4x+2)\sin(x) &= -(4x+2)\cos(x)\Big|_0^4 + 4\int_0^4 \mathrm{d}x\ \cos(x)\\ &= -(4x+2)\cos(x) + 4\sin(x)\Big|_0^4\\ &= -18\cos(4) + 2\cos(0) + 4\sin(4)\\ &= 4(\sin(4) - 3\cos(4)) + 2\end{aligned}$$

d)*

$$\begin{aligned}\int_0^1 \mathrm{d}x\, 2x^3\mathrm{e}^{x^2} &= 2\int_0^1 \mathrm{d}x\, x^2\cdot x\mathrm{e}^{x^2}\\ &= 2\left(x^2\frac{\mathrm{e}^{x^2}}{2}\Big|_0^1 - 2\int_0^1 \mathrm{d}x\ \frac{x\mathrm{e}^{x^2}}{2}\right)\\ &= 2\frac{\mathrm{e}^{x^2}}{2}\left(x^2-1\right)\Big|_0^1\\ &= 1\end{aligned}$$

Aufgabe Partialbruchzerlegung

a)

$$\begin{aligned}\frac{1}{(x+1)(x+2)} &= \frac{a}{x+1} + \frac{b}{x+2} \\ &= \frac{a(x+2)}{(x+1)(x+2)} + \frac{b(x+1)}{(x+1)(x+2)} \\ \Rightarrow \quad 1 &= a(x+2) + b(x+1) \\ &= x(a+b) + (2a+b)\end{aligned}$$

$$\begin{aligned}&\Rightarrow \quad a+b=0 \quad \wedge \quad 2a+b=1 \\ &\Leftrightarrow \quad a=-b \quad \Rightarrow \quad -2b+b=1=-b.\end{aligned}$$

$$\begin{aligned}\int_0^1 \mathrm{d}x \, \frac{1}{(x+1)(x+2)} &= \int_0^1 \mathrm{d}x \, \frac{1}{x+1} - \int_0^1 \mathrm{d}x \, \frac{1}{x+2} \\ &= \log(x+1) - \log(x+2)\Big|_0^1 \\ &= \log\left(\frac{x+1}{x+2}\right)\Big|_0^1 \\ &= \log\left(\frac{2}{3}\right) - \log\left(\frac{1}{2}\right) \\ &= \log\left(\frac{4}{3}\right)\end{aligned}$$

b)

$$\begin{aligned}\frac{x}{(x+1)(x+2)} &= \frac{a}{x+1} + \frac{b}{x+2} \\ &= \frac{a(x+2)}{(x+1)(x+2)} + \frac{b(x+1)}{(x+1)(x+2)} \\ \Rightarrow \quad x &= a(x+2) + b(x+1) \\ &= x(a+b) + (2a+b)\end{aligned}$$

$$\begin{aligned}&\Rightarrow \quad a+b=1 \quad \wedge \quad 2a+b=0 \\ &\Leftrightarrow \quad a=-\frac{b}{2} \\ &\Rightarrow \quad \frac{b}{2}=1 \\ &\Rightarrow \quad b=2 \quad a=-1\end{aligned}$$

$$\begin{aligned}\int_0^1 \mathrm{d}x \,\frac{x}{(x+1)(x+2)} &= 2\int_0^1 \mathrm{d}x\,\frac{1}{x+2} - \int_0^1 \mathrm{d}x\,\frac{1}{x+1}\\ &= 2\log(x+2) - \log(x+1)\Big|_0^1\\ &= \log\left(\frac{(x+2)^2}{x+1}\right)\Big|_0^1\\ &= \log\left(\frac{9}{8}\right)\end{aligned}$$

Aufgaben Substitution

a)

$$\int_{-1}^1 \mathrm{d}x\, 3(3x-1)^4, \quad y := 3x-1 \quad \Rightarrow \quad \mathrm{d}y = 3\,\mathrm{d}x$$

$$\begin{aligned}\int_{-1}^1 \mathrm{d}x\, 3(3x-1)^4 &= \int_{-4}^2 \mathrm{d}y\, y^4\\ &= \frac{y^5}{5}\Big|_{-4}^2\\ &= \frac{1056}{5}\end{aligned}$$

b)

$$\int_a^b \mathrm{d}x\, c\mathrm{e}^{-kx}, \quad y := kx \quad \Rightarrow \quad \mathrm{d}y = k\,\mathrm{d}x$$

$$\begin{aligned}\int_a^b \mathrm{d}x\, c\mathrm{e}^{-kx} &= \frac{c}{k}\int_{ka}^{kb} \mathrm{d}y\, \mathrm{e}^{-y}\\ &= -\frac{c}{k}\mathrm{e}^{-y}\Big|_{ka}^{kb}\\ &= \frac{c}{k}\left(\mathrm{e}^{-ka} - \mathrm{e}^{-kb}\right)\end{aligned}$$

c)*

$$\int_0^1 \mathrm{d}x\,\sqrt{1-x^2}, \quad x = \sin(y) \quad \Rightarrow \quad \mathrm{d}x\; = \cos(y)\,\mathrm{d}y$$

$$
\begin{aligned}
\int_0^1 \mathrm{d}x\,\sqrt{1-x^2} &= \int_{\sin^{-1}(0)}^{\sin^{-1}(1)} \mathrm{d}y\,\sqrt{1-\sin^2(y)}\cos(y) \\
&= \int_0^{\pi/2} \mathrm{d}y\,\sqrt{\cos^2(y)}\cos(y) \\
&= \int_0^{\pi/2} \mathrm{d}y\,\cos^2(y) \\
&= \sin(y)\cos(y)\Big|_0^{\pi/2} + \int_0^{\pi/2} \mathrm{d}y\,\sin^2(y) \\
&= \sin(y)\cos(y)\Big|_0^{\pi/2} + \int_0^{\pi/2} \mathrm{d}y\,(1-\cos^2(y))
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \quad & 2\int_0^{\pi/2} \mathrm{d}y\,\cos^2(y) = \sin(y)\cos(y)\Big|_0^{\pi/2} + \int_0^{\pi/2} \mathrm{d}y \\
\Rightarrow \quad & \int_0^{\pi/2} \mathrm{d}y\,\cos^2(y) = \sin(y)\cos(y)\Big|_0^{\pi/2} + \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}
\end{aligned}
$$

Aufgabe ?

Für $f(x) = g(x)h(x)$ existiert ein solcher Zusammenhang nicht.