Quantum Field Theory (QFT)

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E5 - 129

- Some Literature

[IZ] C. Itzykson, J.-B. Zuber, QFT
1980, McGraw-Hill

[PS] M.E Peskin, D.V. Schroeder, An Introduction to QFT
1995 Westview

[R] L.H. Ryder, QFT
1985 Cambridge Univ. Press

Tutor: Jan Möller

Lecture	Mondays	10 - 12	S2 - 121
	Wednesdays	10 - 12	T2 - 233
Exercises	"	12^{30} - 14	

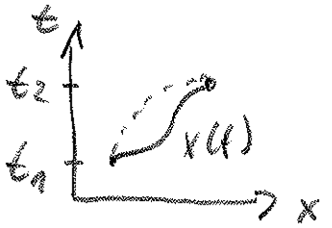
1. Canonical Quantization of scalar fields

1.1. Classical field theory in Lagrange and Hamilton formalism

Reminder: classical particle

- kinetic energy $T = \frac{1}{2} m \dot{x}^2$ $\quad (\dot{x} = \frac{dx}{dt})$
- potential energy $V = V(x)$
- Lagrangian $L = T - V = \frac{1}{2} m \dot{x}^2 - V(x) = L(x, \dot{x})$ ← in general not explicitly t-dependent
- action $S = \int_{t_1}^{t_2} dt\, L(x, \dot{x})$

- Principle of minimal action: class. particle follows path $x(t)$ that minimises the action



Variations

$x(t) \to x'(t) = x(t) + \delta x(t)$

$\delta x(t_1) = 0 = \delta x(t_2)$ boundary conditions

$$\Rightarrow S' = S + \delta S = \int_{t_1}^{t_2} dt\, L(x + \delta x, \dot{x} + \delta \dot{x}) \qquad \left(\delta \dot{x} = \frac{d\, \delta x(t)}{dt} \right)$$

$$\approx S + \int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial \dot{x}} \delta \dot{x} \right\}$$

$$= S + \underbrace{\int_{t_1}^{t_2} dt \frac{d}{dt} \left\{ \frac{\partial L}{\partial \dot{x}} \delta x \right\}}_{=0 \text{ b.c.}} + \underbrace{\int_{t_1}^{t_2} dt \left\{ \frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) \right\} \delta x}_{\text{has to vanish } \forall \delta x \Rightarrow \text{ integrand } \equiv 0}$$

$\Rightarrow$ Euler-Lagrange-eq: $\boxed{\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) = 0} \iff -\frac{\partial V}{\partial x} - m \ddot{x} = 0$

Lagrange formalism $\qquad\qquad$ 2nd order

- Hamilton formalism: (from Lagrangian)
  - canonical momentum $p = \frac{\partial L}{\partial \dot{x}} = m \dot{x}$
  - Hamilton function $H = p \dot{x} - L = \frac{p^2}{2m} + V(x) = H(x, p)$
  - Hamilton eqs $\boxed{\dot{x} = \frac{\partial H}{\partial p} \; ; \; \dot{p} = -\frac{\partial H}{\partial x}}$ 1st order

$\iff \dot{x} = \frac{p}{m} \; , \; \dot{p} = m \ddot{x} = -\frac{\partial V}{\partial x}$

- principle of least action

$$S' = S + \delta S = \int_\Omega dx^4\, \mathcal{L}(\phi', \partial_\mu \phi') \quad , \quad \phi \to \phi' = \phi + \delta\phi$$

$$\approx S + \int_\Omega dx^4 \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta \partial_\mu \phi \right\} \quad , \quad \partial_\mu \phi' = \partial_\mu \phi + \partial_\mu(\delta\phi)$$

$$= S + \underbrace{\int_\Omega dx^4\, \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right\}} + \underbrace{\int_\Omega dx^4 \left\{ \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] \right\} \delta\phi}$$

$$= \int_{\partial\Omega} ds\, n_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right\} \qquad \Rightarrow \text{ zero } \forall \delta\phi \text{ , integrand } 0$$

vanishes if variation $\delta\phi = 0$
on surface S

$\Rightarrow$ Euler - Lagrange - eq $\quad \boxed{\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right] = 0}$

Example: $\mathcal{L} = \frac{1}{2} \partial^\mu \phi\, \partial_\mu \phi - \frac{1}{2} m^2 \phi^2 \qquad$ 1 scalar field $\phi(x^\mu)$

(use g)

$$\Rightarrow \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi \, , \quad \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi \quad \Rightarrow \quad - \boxed{(\partial_\mu \partial^\mu \phi + m^2 \phi) = 0}$$

Klein-Gordon eq.

Hamilton formalism

- canonical momentum $\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial \mathcal{L}}{\partial(\partial_0 \phi)} = \partial^0 \phi \quad (= \partial_0 \phi)$
- Hamilton density from Legendre trafo:

$$\boxed{\mathcal{H}(\phi, \pi) = \pi \partial_0 \phi - \mathcal{L} = \frac{1}{2} \pi(x)^2 + \frac{1}{2} (\partial_i \phi)^2 + \frac{1}{2} m^2 \phi^2}$$

$\Rightarrow$ Hamilton equations can be generalised (try yourself!)

Advantage of Lagrange: formulation of Noether-Theorem [Emmy Noether 1918]

$$\exists \text{ Symmetry} \xleftrightarrow{1-1} \exists \text{ conserved quantity}$$

②

- can be expressed with Poisson brackets (later),
  x, p are conjugate variables

* both Lagrange and Hamilton formalism are equivalent to Newton's formalism. But both L and H are easier to quantize.

- classical fields (relativistic)

replace degrees of freedom (d.o.f.) $x \to \phi(x^\mu)$

coordinates $t \to x^\mu = (x^0, x^i) = (ct, \vec{x})$
$i = 1,2,3$

relativistic notation

metric $\eta^{\mu\nu} = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix} = \eta_{\mu\nu}$, $x_\mu = \eta_{\mu\nu} x^\nu$, $x^\mu = \eta^{\mu\nu} x_\nu$

with Einstein summation convention $\hat{=} \sum_{\nu=0}^{3} \eta_{\mu\nu} x^\nu$
over indices appearing twice
(greek letters $\mu, \nu = 0,1,2,3$, latin letters $i,j = 1,2,3$)

$\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\partial^\mu = \frac{\partial}{\partial x_\mu}$, $\partial_\mu x^\nu = \delta_\mu^{\ \nu} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

- contraction of 1 upper and 1 lower index has no index,
  scalar product e.g. $x_\nu x^\nu$, $\partial_\mu x^\mu$

Lagrange formalism on classical fields

- Lagrange density $\mathcal{L}(\phi, \partial_\mu \phi)$, in general no expl. x^μ-dep.
- Lagrange function $L[\phi, \partial_\mu \phi] = \int_V d\vec{x}^3\, \mathcal{L}(\phi, \partial_\mu \phi)$
  V – 3 volume
- action $S = \int_{t_1}^{t_2} dt\, L[\phi, \partial_\mu \phi]$
  $= \int_\Omega dx^4\, \mathcal{L}(\phi, \partial_\mu \phi)$, $\Omega = [t_1, t_2] \times V$
  4 volume

Recall: the Euler-Lagrange eq. followed from $\delta S = 0$ varying ϕ with <u>fixed</u> boundary conditions on $\partial\Omega$

* There are other trafos that leave S invariant or have $\delta S = 0$

<u>examples</u>

- space-time coordinate trafos, e.g. translations

  $\Rightarrow$ exercise 1.1

  conserved quantity: $\partial_\mu T^{\mu\nu} = 0$

  $T^{\mu\nu}$ energy-momentum tensor,

  with $T^{00} = \mathcal{H}$

- internal symmetries

Further details are in exercise 1.

Derivation of Noether's Theorem (for scalar fields)

Simultaneous trafo of i internal label

space-time coord $\quad x^\mu \to x^{\mu\prime} = x^\mu + \delta x^\mu \qquad \delta x^\mu = X^\mu{}_i(x)\,\delta\omega^i$

fields $\quad \phi^a(x) \to \phi^{a\prime}(x') = \phi^a(x) + \delta\phi^a(x) \quad , \quad \delta\phi^a(x) = \Phi_i{}^a(x)\,\delta\omega^i$

generators $\delta\omega^i$, x^μ-indep

we assume that this is a symmetry:

$$\Delta S' = \int_\Delta d^4x' \, \mathcal{L}\left(\phi^{a\prime}(x'), \partial'_\mu \phi^{a\prime}(x')\right) = \Delta S + O((\delta\omega^i)^2) \quad \text{for small 4-vol } \Delta$$

for $\delta\omega^i$ x^μ-indep and $\phi^a(x)$ satisfying Euler-Lagrange (on-shell).

The vanishing of $O(\delta\omega^i)$ will give the conserved quantity

Building blocks of the variation

- Jacobi-det $\quad d^4x' = \det\left[\frac{\partial x^{\mu\prime}}{\partial x^\nu}\right] d^4x = \det\left[\delta^\mu{}_\nu + \partial_\nu(X^\mu{}_i\,\delta\omega^i)\right] d^4x$

Taylor expand $\det(1+xA) = \exp[\mathrm{Tr}\log(1+xA)] \Rightarrow \det[1+xA] = 1 + x\,\mathrm{Tr}[A] + O(x^2)$ $\quad = \left[1 + \partial_\mu X^\mu{}_i\,\delta\omega^i\right] d^4x + O(\delta\omega^{i2})$

- Lagrangian $\quad \mathcal{L}(\phi^{a\prime}, \partial'_\mu\phi^{a\prime}) = \mathcal{L}(\phi^a, \partial_\mu\phi^a) + \frac{\partial\mathcal{L}}{\partial\phi^a}\delta\phi^a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\delta(\partial_\mu\phi^a) + O(\delta\omega^{i2})$

with
$$\begin{aligned}
\delta(\partial_\mu\phi^a) &= \partial'_\mu\phi^{a\prime} - \partial_\mu\phi^a = \frac{\partial x^\mu}{\partial x^{\nu\prime}}\partial_\nu\phi^{a\prime} - \partial_\mu\phi^a \\
&= \left[\delta^\mu{}_\nu - \partial_\mu(X^\nu{}_i\,\delta\omega^i)\right]\partial_\nu\phi^{a\prime} - \partial_\mu\phi^a + O(\delta\omega^{i2}) \\
&= \partial_\mu(\phi^{a\prime} - \phi^a) - \partial_\mu X^\nu{}_i\,\delta\omega^i\,\partial_\nu\phi^a + O(\delta\omega^{i2}) \\
&= \left(\partial_\mu\Phi_i{}^a - \partial_\mu X^\nu{}_i\,\partial_\nu\phi^a\right)\delta\omega^i + O(\delta\omega^{i2})
\end{aligned}$$

$$\Rightarrow \Delta S' \overset{\text{E-L eq}}{=} \Delta S + \int_\Delta d^4x \left\{ \mathcal{L}\,\delta^\mu{}_\nu\,\partial_\mu X^\nu{}_i + \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\right]\Phi^a{}_i + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\left(\partial_\mu\Phi^a{}_i - \partial_\mu X^\nu{}_i\,\partial_\nu\phi^a\right)\right\}\delta\omega^i + O(\delta\omega^{i2})$$

we have
$$\begin{aligned}
\partial_\mu\left[\mathcal{L}\,\delta^\mu{}_\nu - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\partial_\nu\phi^a\right] &= \partial_\nu\mathcal{L} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\right]\partial_\nu\phi^a - \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\partial_\mu\partial_\nu\phi^a \\
&= \partial_\nu\phi^a\frac{\partial\mathcal{L}}{\partial\phi^a} + \partial_\nu\partial_\mu\phi^a\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)} - \partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)}\right]\partial_\nu\phi^a - \partial_\nu\partial_\mu\phi^a\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^a)} \overset{\text{E.L. eq.}}{=} 0
\end{aligned}$$

$$\Rightarrow \quad \Delta S' = \Delta S + \int_\Delta d^4x \{ \partial_\mu j^\mu{}_i \} \, \delta\omega^i + O(\delta\omega^{i2})$$

trafo = symmetry $\Rightarrow$ $\{ \partial_\mu j^\mu{}_i \} = 0 \quad \forall i$

with $$\underline{j^\mu{}_i = \left[\mathcal{L} \delta^\mu{}_\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \partial_\nu \phi^a \right] X^\nu{}_i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \Phi^a{}_i}$$

conserved Noether - current for symmetry trafo

The corresponding conserved charge(s)

$$\underline{Q_i = \int_V d^3x \, j^0_i} \quad : \quad \partial^0 Q_i = 0$$

(see ex 1.2 for details)

reading: [PS ch 2.2]

[IZ ch 1-1-1, 1-2]