# 1.3. Canonical quantisation of a free scalar field

Start: p.3 $H = \int d\vec{x}^3 \left\{ \frac{1}{2} \pi(x^0, \vec{x})^2 + \frac{1}{2}\left(\partial_i \phi(x^0, \vec{x})\right)^2 + \frac{1}{2} m^2 \phi(x^0, \vec{x})^2 \right\}$

$$\pi(x^0, \vec{x}) = \partial_0 \phi(x^0, \vec{x})$$

3 steps as for classical particle:

i) fields → op: $\phi \to \hat{\phi}$, $\pi \to \hat{\pi}$, $H \to \hat{H}$

ii) canon. var.: equal time commutators ($\hat{=}$ normalisation)

$$[\hat{\phi}(x^0, \vec{x}), \hat{\phi}(x^0, \vec{y})] = 0 = [\hat{\pi}(x^0, \vec{x}), \hat{\pi}(x^0, \vec{y})]$$

$$[\hat{\phi}(x^0, \vec{x}), \hat{\pi}(x^0, \vec{y})] \equiv i\, \delta^{(3)}(\vec{x} - \vec{y})$$

iii) <u>dynamics from Heisenberg e.o.m.</u>

$$i \partial_0 \hat{\phi}(x^0, \vec{x}) \equiv [\hat{\phi}(x^0, \vec{x}), \hat{H}]$$

$$i \partial_0 \hat{\pi}(x^0, \vec{x}) \equiv [\hat{\pi}(x^0, \vec{x}), \hat{H}]$$

where $\hat{H} = \int d\vec{x}^3 \left\{ \frac{1}{2} \hat{\pi}^2 + \frac{1}{2} \hat{\phi} [-\vec{\nabla}^2 + m^2] \hat{\phi} \right\}$ (int. by parts):

we get (check!)

$$\underline{[\hat{\phi}(x^0, \vec{x}), \hat{H}]} = \int d\vec{y}^3 \, \hat{\pi}(x^0, \vec{y}) [\hat{\phi}(x^0, \vec{x}), \hat{\pi}(x^0, \vec{y})] = \underline{i \hat{\pi}(x^0, \vec{x})}$$

$$\underline{[\hat{\pi}(x^0, \vec{x}), \hat{H}]} = \int d\vec{y}^3 \, [\hat{\pi}(x^0, \vec{x}), \hat{\phi}(x^0, \vec{y})] (-\vec{\nabla}_y^2 + m^2) \hat{\phi}(x^0, \vec{y})$$

$$= \underline{-i(-\vec{\nabla}_x^2 + m^2) \hat{\phi}(x^0, \vec{x})}$$

$$\Rightarrow \partial_0^2 \hat{\phi}(x^0, \vec{x}) = \partial_0 \hat{\pi}(x^0, \vec{x}) = (\vec{\nabla}^2 - m^2) \hat{\phi}(x^0, \vec{x})$$

$$\Leftrightarrow \boxed{[\partial_0^2 - \vec{\nabla}^2 + m^2] \hat{\phi}(x^0, \vec{x}) = 0} \quad \text{Klein-Gordon eq.}$$

• same e.o.m. as classical field!

- the Klein-Gordon eq. can be solved in terms of plane waves: (12)

Ansatz: $\hat{\phi}(x^0,\vec{x}) = \hat{N}(p)\, e^{-i p_\mu x^\mu}$

$\Rightarrow (-p_0^2 + \vec{p}^{\,2} + m^2)\hat{N}(p) = 0$ , let $E_{\vec{p}} = \sqrt{\vec{p}^{\,2}+m^2} \Rightarrow p_0 = \pm E_{\vec{p}}$

$(E_{-\vec{p}} = E_{+\vec{p}})$

- the most general solution is a superposition:

$$\hat{\phi}(x^0,\vec{x}) = \int d^3\vec{p}\,\Big[\underbrace{\hat{N}_+(\vec{p})\,e^{-iE_{\vec{p}}x^0 + i\vec{p}\vec{x}}}_{\uparrow\text{ pos. energy}} + \underbrace{\hat{N}_-(\vec{p})\,e^{+iE_{\vec{p}}x^0 - \dots}}_{\uparrow\text{ neg. ''}}\Big\}$$

$$\overset{\substack{\vec{p}\to-\vec{p}\\ \text{in 2nd term}}}{=} \int d^3\vec{p}\,\Big\{\hat{N}_+(\vec{p})\,e^{-ip\cdot x} + \hat{N}_-(-\vec{p})\,e^{+ip\cdot x}\Big\}$$

with scalar product $p\cdot x = p_\mu x^\mu = p^\mu x_\mu$ , $p^\mu = (E_{\vec{p}},\vec{p})$

- in analogy to HO we write

$$\hat{N}_+(\vec{p}) = \frac{1}{\sqrt{(2\pi)^3\, 2E_{\vec{p}}}}\,\hat{a}_{\vec{p}}\ ,\quad \hat{N}_-(-\vec{p}) = \frac{1}{\sqrt{(2\pi)^3\, 2E_{\vec{p}}}}\,\hat{a}^\dagger_{\vec{p}} \qquad (\hbar\omega = E_{\vec{p}})$$

- to satisfy the com. $[\hat{\phi}(x^0,\vec{x}), \partial_0\hat{\phi}(x^0,\vec{y})] = i\,\delta^{(3)}(\vec{x}-\vec{y})$ we get for $\hat{a}$'s

$$\boxed{\begin{aligned}[\hat{a}_{\vec{p}},\hat{a}_{\vec{q}}] = 0 = [\hat{a}^\dagger_{\vec{p}},\hat{a}^\dagger_{\vec{q}}] \quad & \text{I, II}\\ [\hat{a}_{\vec{p}},\hat{a}^\dagger_{\vec{q}}] = \delta^{(3)}(\vec{p}-\vec{q}) \quad & \text{III}\end{aligned}}$$

check: $$\boxed{\hat{\phi}(x^0,\vec{x}) = \underbrace{\int\frac{d^3\vec{p}}{\sqrt{(2\pi)^3\,2E_{\vec{p}}}}}_{\equiv \int_{\vec{p}}}\Big\{\hat{a}_{\vec{p}}\,e^{-ip\cdot x} + \hat{a}^\dagger_{\vec{p}}\,e^{ip\cdot x}\Big\}}$$

$$\Rightarrow \partial_0\hat{\phi}(x^0,\vec{x}) = \int_{\vec{p}}\Big\{-iE_{\vec{p}}\,\hat{a}_{\vec{p}}\,e^{-ip\cdot x} + iE_{\vec{p}}\,\hat{a}^\dagger_{\vec{p}}\,e^{ip\cdot x}\Big\}$$

$$\Rightarrow \underset{\text{def } y=(x^0,\vec{y})}{[\hat{\phi}(x^0,\vec{x}), \partial_0\hat{\phi}(x^0,\vec{y})]} \overset{\text{I, II}}{=} \int_{\vec{p}}\int_{\vec{q}}\Big\{iE_{\vec{q}}\underbrace{[\hat{a}_{\vec{p}},\hat{a}^\dagger_{\vec{q}}]}_{\overset{\text{III}}{=}\ \delta^{(3)}(\vec{p}-\vec{q})}e^{-ipx+iqy} - iE_{\vec{q}}\underbrace{[\hat{a}^\dagger_{\vec{p}},\hat{a}_{\vec{q}}]}_{=-\delta^{(3)}(\vec{p}-\vec{q})}e^{+ipx-iq\cdot y}\Big\}$$

$$= \int\frac{d^3\vec{p}}{(2\pi)^3\,2E_{\vec{p}}}\, iE_{\vec{p}}\Big\{e^{i\vec{p}(\vec{x}-\vec{y})} + e^{-i\vec{p}(\vec{x}-\vec{y})}\Big\}$$

$$\overset{\substack{\vec{p}\to-\vec{p}\\ \text{in 2nd term}}}{=} i\int\frac{d^3\vec{p}}{(2\pi)^3}\,e^{i\vec{p}(\vec{x}-\vec{y})} = i\,\delta^{(3)}(\vec{x}-\vec{y}) \qquad \checkmark$$

- $[\hat{\phi},\hat{\phi}] = [\hat{\pi},\hat{\pi}] = 0$ hold too to continue the HO-analogy we express $\hat{H}$ in terms of $\hat{a}_{\vec{p}}, \hat{a}^+_{\vec{p}}$:

- insert $\hat{\phi} = \int_{\vec{p}} \{ \hat{a}_{\vec{p}}\, e^{-ip\cdot x} + \hat{a}^+_{\vec{p}}\, e^{ip\cdot x} \}$ into $\hat{H}$

$$
\begin{aligned}
\hat{H} &= \int d\vec{x}^3 \{ \tfrac{1}{2}\partial_0\hat{\phi}\,\partial_0\hat{\phi} + \tfrac{1}{2}\partial_i\hat{\phi}\,\partial_i\hat{\phi} + \tfrac{1}{2}m^2\hat{\phi}\hat{\phi} \} \\
&= \tfrac{1}{2}\int d\vec{x}^3 \int_{\vec{p}}\int_{\vec{q}} \Big\{ [-iE_{\vec{p}}\hat{a}_{\vec{p}}e^{-ipx} + iE_{\vec{p}}\hat{a}^+_{\vec{p}}e^{ipx}][-iE_{\vec{q}}\hat{a}_{\vec{q}}e^{-iqx} + iE_{\vec{q}}\hat{a}^+_{\vec{q}}e^{iqx}] \\
&\qquad + [ip_i\hat{a}_{\vec{p}}e^{-ipx} - ip_i\hat{a}^+_{\vec{p}}e^{ipx}][iq_i\hat{a}_{\vec{q}}e^{-iqx} - iq_i\hat{a}^+_{\vec{q}}e^{iqx}] \\
&\qquad + m^2[\hat{a}_{\vec{p}}e^{-ipx} + \hat{a}^+_{\vec{p}}e^{ipx}][\hat{a}_{\vec{q}}e^{-iq\cdot x} + \hat{a}^+_{\vec{q}}e^{iq\cdot x}] \Big\} \\
&= \tfrac{1}{2}\int d\vec{x}^3 \int_{\vec{p}}\int_{\vec{q}} \Big\{ [-E_{\vec{p}}E_{\vec{q}} - \vec{p}\vec{q} + m^2]\hat{a}_{\vec{p}}\hat{a}_{\vec{q}}e^{-i(p+q)\cdot x} + [E_{\vec{p}}E_{\vec{q}} + \vec{p}\vec{q} + m^2]\hat{a}_{\vec{p}}\hat{a}^+_{\vec{q}}e^{-i(p-q)\cdot x} \\
&\qquad + [E_{\vec{p}}E_{\vec{q}} + \vec{p}\vec{q} + m^2]\hat{a}^+_{\vec{p}}\hat{a}_{\vec{q}}e^{i(p-q)\cdot x} + [-E_{\vec{p}}E_{\vec{q}} - \vec{p}\vec{q} + m^2]\hat{a}^+_{\vec{p}}\hat{a}^+_{\vec{q}}e^{i(p+q)\cdot x} \Big\}
\end{aligned}
$$

use $\int \frac{d\vec{x}^3}{(2\pi)^3} e^{i\vec{k}\vec{x}} = \delta^{(3)}(\vec{k})$

$$
\begin{aligned}
&= \frac{1}{2}\int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}}(2\pi)^3 \Big\{ \underbrace{[-E^2_{\vec{p}} + \vec{p}^2 + m^2]}_{=0 \text{ e.o.m}}\hat{a}_{\vec{p}}\hat{a}_{-\vec{p}}e^{-2iE_{\vec{p}}x^0} + [E^2_{\vec{p}} + \vec{p}^2 + m^2]\hat{a}_{\vec{p}}\hat{a}^+_{\vec{p}} \\
&\qquad + [E^2_{\vec{p}} + \vec{p}^2 + m^2]\hat{a}^+_{\vec{p}}\hat{a}_{\vec{p}} + \underbrace{[-E^2_{\vec{p}} + \vec{p}^2 + m^2]}_{=0 \text{ e.o.m}}\hat{a}^+_{\vec{p}}\hat{a}^+_{-\vec{p}}e^{2iE_{\vec{p}}x^0} \Big\}
\end{aligned}
$$

$$
\boxed{\hat{H} = \int d^3\vec{p}\, \tfrac{1}{2}E_{\vec{p}}\,(\hat{a}_{\vec{p}}\hat{a}^+_{\vec{p}} + \hat{a}^+_{\vec{p}}\hat{a}_{\vec{p}})}
$$

compare to HO, Exercise 2

- this is a Hermitian operator $\Rightarrow$ its eigenvalues are real

<u>commutator relations</u>: (use $[AB,C] = A[B,C] + [A,C]B$)

$$
\begin{aligned}
[\hat{H}, \hat{a}_{\vec{k}}] &= \int d\vec{p}^3 E_{\vec{p}}\tfrac{1}{2}\Big\{ \hat{a}_{\vec{p}}\underbrace{[a^+_{\vec{p}}, \hat{a}_{\vec{k}}]}_{=-\delta^{(3)}(\vec{p}-\vec{k})} + \underbrace{[\hat{a}_{\vec{p}}, \hat{a}_{\vec{k}}]}_{=0}\hat{a}^+_{\vec{p}} + \hat{a}^+_{\vec{p}}\underbrace{[\hat{a}_{\vec{p}}, \hat{a}_{\vec{k}}]}_{=0} + \underbrace{[\hat{a}^+_{\vec{p}}, \hat{a}_{\vec{k}}]}_{=-\delta^{(3)}(\vec{p}-\vec{k})}\hat{a}_{\vec{p}} \Big\} \\
&= -E_{\vec{k}}\,\hat{a}_{\vec{k}}
\end{aligned}
$$

$$
\begin{aligned}
[\hat{H}, \hat{a}^+_{\vec{k}}] &= \int d\vec{p}^3 E_{\vec{p}}\tfrac{1}{2}\Big\{ \hat{a}_{\vec{p}}\underbrace{[\hat{a}^+_{\vec{p}}, \hat{a}^+_{\vec{k}}]}_{=0} + [\hat{a}_{\vec{p}}, \hat{a}^+_{\vec{k}}]\hat{a}^+_{\vec{p}} + \hat{a}^+_{\vec{p}}[\hat{a}_{\vec{p}}, \hat{a}^+_{\vec{k}}] + \underbrace{[\hat{a}^+_{\vec{p}}, \hat{a}^+_{\vec{k}}]}_{=0}\hat{a}_{\vec{p}} \Big\} \\
&= +E_{\vec{k}}\,\hat{a}^+_{\vec{k}}
\end{aligned}
$$

## Energy eigenstates

Let $|\Psi\rangle$ be an eigenstate of $\hat{H}$ : $\hat{H}|\Psi\rangle = E_\Psi|\Psi\rangle$, eigenvalue $E_\Psi$ = energy

$$\Rightarrow \quad \hat{H}\hat{a}_{\vec{k}}|\Psi\rangle = \left([\hat{H},\hat{a}_{\vec{k}}] + \hat{a}_{\vec{k}}\hat{H}\right)|\Psi\rangle = (-E_{\vec{k}} + E_\Psi)\,\hat{a}_{\vec{k}}|\Psi\rangle$$

so $\underline{\hat{a}_{\vec{k}}|\Psi\rangle}$ is an eigenstate with energy $\underline{(-E_{\vec{k}} + E_\Psi)}$

analogously $\underline{\hat{a}^\dagger_{\vec{k}}|\Psi\rangle}$ ——— " ——— $\underline{(+E_{\vec{k}} + E_\Psi)}$

As for the HO we assume the existence of a <u>vacuum state $|0\rangle$</u>
with minimal energy : $\forall \vec{k}\ \hat{a}_{\vec{k}}|0\rangle = 0$ (and norm $\langle 0|0\rangle = 1$)
(( a new eigenstate with lower energy cannot be created by applying $\hat{a}_{\vec{k}}$ ))
All other states are then of the form:

$$\hat{a}^\dagger_{\vec{k}}|0\rangle \equiv |\vec{k}\rangle \qquad \text{single particle states}$$

$$\hat{a}^\dagger_{\vec{k}_1}\hat{a}^\dagger_{\vec{k}_2}\ldots\hat{a}^\dagger_{\vec{k}_n}|0\rangle = |\vec{k}_1,\vec{k}_2,\ldots\vec{k}_n\rangle \qquad \text{multiple particle states}$$

They constitute the Fock-space of states, with

<u>norm</u>: $\underline{\langle\vec{p}|\vec{q}\rangle} = \langle 0|\hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{q}}|0\rangle = \langle 0|\{[\hat{a}_{\vec{p}},\hat{a}^\dagger_{\vec{q}}] + \hat{a}^\dagger_{\vec{q}}\hat{a}_{\vec{p}}\}|0\rangle = \underline{\delta^3(\vec{p}-\vec{q})}$

Q: <u>What is the vacuum energy?</u>

$$\langle 0|\hat{H}|0\rangle = \int d^3\vec{p}\, E_{\vec{p}}\, \tfrac{1}{2}\langle 0|\underbrace{\hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{p}}}_{=\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}} + \delta^{(3)}(\vec{p}-\vec{p})} + \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}|0\rangle = \int d^3\vec{p}\,\frac{E_{\vec{p}}}{2}\,\delta^{(3)}(0) = \text{"}\infty\text{"}$$

Although we may always refer to energies as to differences to the vacuum energy this "$\infty$" can be removed by <u>normal ordering</u>:

$$:\hat{O}: \equiv \text{" } \hat{O} \text{ with all anihilation operators } \hat{a}_{\vec{p}} \text{ moved to the right "}$$

$$\Rightarrow \; :\hat{H}: = \int d^3\vec{p}\,\frac{E_{\vec{p}}}{2}\left(\hat{a}_{\vec{p}}\hat{a}^\dagger_{\vec{p}} + \hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}\right) = \int d^3\vec{p}\, E_{\vec{p}}\,\hat{a}^\dagger_{\vec{p}}\hat{a}_{\vec{p}}$$

has $\langle 0|:\hat{H}:|0\rangle = 0$ .

reading: [PS] ch 2.3, [IZ] ch 3.1.1, 3.1.2