1.4. Propagators - 2 point functions

We will now consider several 2-point-Green-functions or propagators. They play an important rôle in QFT.

(reminder Green's fnct. $G(x,y)$ for diff. op. D_x : $D_x G(\vec{x},\vec{y}) = \delta^{(3)}(x-y)$,
e.g. for $D_x = \Delta$ → construct solutions for the Poisson eq.
$-\Delta\phi(\vec{x}) = 4\pi\rho(\vec{x})$ for a given chargedensity $\rho(\vec{x})$: $\int d^3\vec{y}\, G(\vec{x},\vec{y})(-4\pi\rho(\vec{y}))$ solves)

back to QFT:

2-point funct $\boxed{W(x,y) \equiv \langle 0|\hat{\phi}(x)\hat{\phi}(y)|0\rangle}$ Wightman-function

If $x^0 = y^0$ it holds $W(x,y) = W(y,x)$ as equal time $\hat{\phi}$ commute.

In general $W(x,y) \neq W(y,x)$ → def. many nontrivial quantities:

$$* \quad S(x,y) \equiv \langle 0|\tfrac{1}{2}[\hat{\phi}(x),\hat{\phi}(y)]|0\rangle$$

$$\Delta(x,y) \equiv \langle 0|\tfrac{1}{2}\{\hat{\phi}(x),\hat{\phi}(y)\}|0\rangle \quad , \{A,B\} = AB + BA \text{ anticommutator}$$

$$G_R(x,y) \equiv i\langle 0|[\hat{\phi}(x),\hat{\phi}(y)]\Theta(x^0-y^0)|0\rangle \quad \text{retarded Green funct}$$

$$G_A(x,y) \equiv -i\langle 0|[\hat{\phi}(x),\hat{\phi}(y)]\Theta(y^0-x^0)|0\rangle \quad \text{advanced " "}$$

$$* \quad G_F(x,y) \equiv \langle 0|\hat{\phi}(x)\hat{\phi}(y)\Theta(x^0-y^0) + \hat{\phi}(y)\hat{\phi}(x)\Theta(y^0-x^0)|0\rangle$$

$$\equiv \langle 0|T\{\hat{\phi}(x)\hat{\phi}(y)\}|0\rangle \quad \text{time-ordered Green funct}$$

Feynman propagator

define $x^0 \equiv -i\tilde{x}^0$, $y^0 \equiv -i\tilde{y}^0$ ⟺ $\tilde{x}^0 = ix^0$, $\tilde{y}^0 = iy^0$, $\tilde{x}^0, \tilde{y}^0 \in \mathbb{R}$ Euclidean times, $\tilde{x} = (\tilde{x}^0, \vec{x})$

$$* \quad G_E(x,y) \equiv \langle 0|\hat{\phi}(\tilde{x})\hat{\phi}(\tilde{y})\Theta(\tilde{x}^0-\tilde{y}^0) + \hat{\phi}(\tilde{y})\hat{\phi}(\tilde{x})\Theta(\tilde{y}^0-\tilde{x}^0)|0\rangle$$

Euclidean Green funct.
Schwinger propagator

Functions * will be most important for us.

We now compute $S(x,y)$ and then relate to the other two * funct's.

$$g(x,y) = \langle 0|\, \tfrac{1}{2}[\hat\phi(x), \hat\phi(y)]\, |0\rangle$$

$$= \tfrac{1}{2} \int_{\vec p}\int_{\vec q} \langle 0|\, [\hat a_{\vec p} e^{-ipx} + \hat a^\dagger_{\vec p} e^{ipx}, \hat a_{\vec q} e^{-iqx} + \hat a^\dagger_{\vec q} e^{iqx}]\, |0\rangle$$

$$= \tfrac{1}{2} \int_{\vec p}\int_{\vec q} \langle 0|\, [\hat a_{\vec p}, \hat a^\dagger_{\vec q}] e^{-ipx+iqy} + [\hat a^\dagger_{\vec p}, \hat a_{\vec q}] e^{ipx-iqy}\, |0\rangle$$

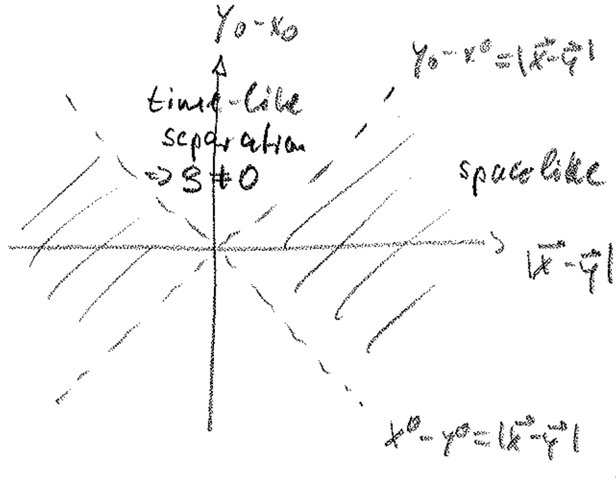
$$= \tfrac{1}{2} \int \frac{d^3\vec p}{(2\pi)^3\, 2E_{\vec p}} \left\{ e^{iE_{\vec p}(y^0-x^0) + i\vec p(\vec x-\vec y)} - e^{iE_{\vec p}(x^0-y^0) + i\vec p(\vec y-\vec x)} \right\} \quad \vec p \to -\vec p$$

$$= \int \frac{d^3\vec p}{(2\pi)^3\, 2E_{\vec p}} e^{i\vec p(\vec x-\vec y)}\, \tfrac{1}{2}\left(e^{iE_{\vec p}(y^0-x^0)} - e^{-iE_{\vec p}(y^0-x^0)}\right) = \int \frac{d^3\vec p}{(2\pi)^3\, 2E_{\vec p}} e^{i\vec p(\vec x-\vec y)}\, i\sin(E_{\vec p}(y^0-x^0))$$

It holds:

- $g(x,y) = 0$ for $x^0 = y^0$
- $g(x,y) \neq 0$ in general
- $\underline{\partial_{y^0}\, g(x,y)\big|_{x^0=y^0}} = \int \frac{d^3\vec p}{(2\pi)^3\, 2E_{\vec p}} e^{i\vec p(\vec x-\vec y)}\, iE_{\vec p}\cdot 1 = \underline{\frac{i}{2}\, \delta^{(3)}(\vec x-\vec y)}$

Causality?



The vanishing of g in non causal region
and non-" in " " can be shown [PS] pp. 27-29

Def spectral function $\tilde g$

$$g(x,y) = \int \frac{d^4p}{(2\pi)^4} e^{ip(y-x)} \left\{ \frac{\pi}{2E_{\vec p}} \left[\delta(p^0-E_{\vec p}) - \delta(p^0+E_{\vec p})\right] \right\}$$

$\Rightarrow \tilde g(p_0,\vec p) \equiv \frac{\pi}{2p_0}\left[\delta(p^0-E_{\vec p}) + \delta(p_0+E_{\vec p})\right]$ Fourrier trafo of $g(x,y)$

Determination of $G_F(x,y)$ Feynman prop

$$W(x,y) = \langle 0|\hat\phi(x)\hat\phi(y)|0\rangle = \int_{\vec p}\int_{\vec q} \langle 0|(\hat a_{\vec p} e^{-ip\cdot x} + \hat a^\dagger_{\vec p} e^{ipx})(\hat a_{\vec q} e^{-iqy} + \hat a^\dagger_{\vec q} e^{iqy})|0\rangle$$

$$= \int_{\vec p}\int_{\vec q} \langle 0|\hat a_{\vec p}\hat a^\dagger_{\vec q} e^{iqy-ipx}|0\rangle \quad , \hat a_{\vec p}\hat a^\dagger_{\vec q} = \hat a^\dagger_{\vec q}\hat a_{\vec p} + [\hat a_{\vec p}, \hat a^\dagger_{\vec q}]$$

$$= \int \frac{d^3\vec p}{(2\pi)^3 2E_{\vec p}} e^{iE_{\vec p}(y^0-x^0) - i\vec p(\vec y-\vec x)}$$

$$\Rightarrow G_F(x,y) = \int \frac{d^3\vec p}{(2\pi)^3 2E_{\vec p}} \left\{\Theta(x^0-y^0) e^{iE_{\vec p}(y^0-x^0)} + \Theta(y^0-x^0) e^{iE_{\vec p}(x^0-y^0)}\right\} e^{-i\vec p(\vec y-\vec x)}$$

after $\vec p \to -\vec p$ in 2nd term

it holds (→ ex 3.1)
$$\lim_{\varepsilon\to 0^+} \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{e^{ik\tau}}{k^2-E^2+i\varepsilon} = -\frac{i}{2E}\left\{\Theta(-\tau)e^{iE\tau} + \Theta(\tau)e^{-iE\tau}\right\}$$

calling $k=p^0$ we have
$$\underline{G_F(x,y)} = i\int \frac{d^3\vec p}{(2\pi)^3} e^{-i\vec p(\vec y-\vec x)} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{e^{ip^0(y^0-x^0)}}{(p^0)^2 - E_{\vec p}^2 + i0^+}$$

$$= \underline{\int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i0^+} e^{ip\cdot(x-y)}} \quad \text{where } p^2 = p\cdot p = (p^0)^2 - \vec p^{\,2}$$

(check: this <u>is</u> a Green function for the scalar field satisfying KG eq.

$$(\underset{\uparrow \text{in } x}{\partial^\mu\partial_\mu} + m^2) G_F(x,y) = (\Box_x + m^2) G_F(x,y) = -i\,\delta^{(4)}(x-y)$$
)

Determination of $G_E(x,y)$ Schwinger prop.

$$G_E(x,y) = \int \frac{d^3\vec p}{(2\pi)^3 2E_{\vec p}} \left\{\Theta(x^0-y^0) e^{+E_{\vec p}(y^0-x^0)} + \Theta(y^0-x^0) e^{E_{\vec p}(x^0-y^0)}\right\} e^{-i\vec p(\vec y-\vec x)}$$

it holds (ex 3.1)
$$\int_{-\infty}^{\infty} \frac{d\tilde k}{2\pi} \frac{e^{i\tilde k\tau}}{\tilde k^2+E^2} = \frac{1}{2E}\left\{\Theta(-\tau)e^{E\tau} + \Theta(\tau)e^{-E\tau}\right\} \quad , \text{call } \tilde p^0 = \tilde k$$

$$\Rightarrow G_E(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{(\tilde p^0)^2 + \vec p^{\,2} + m^2} e^{i\tilde p^0(y^0-x^0) - i\vec p(\vec y-\vec x)} \quad , \underline{P} = (\tilde p^0, \vec p)$$

$$\overset{\vec p\to-\vec p}{=} \underline{\int \frac{d^4\underline{P}}{(2\pi)^4} \frac{1}{\underline{P}^2+m^2} e^{i\underline{P}\cdot(y-x)}}$$

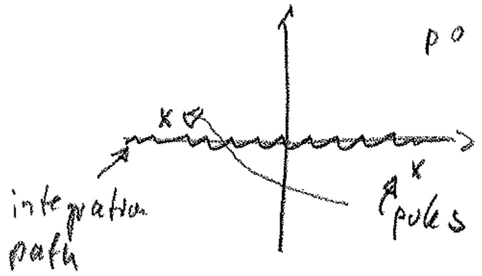
↑ Euclidean scalar products $\underline{P}\cdot\underline{P} = (\tilde p^0)^2 + \vec p^{\,2}$

Summary

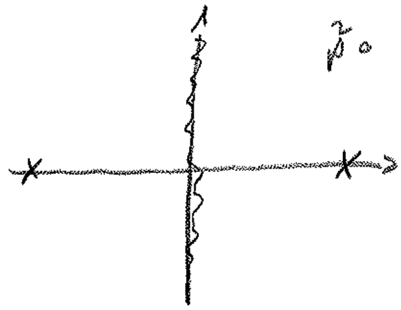
1.) The simplest propagator is the *Euclidean Schwinger prop*, no minus, no ε, no "i".
Later we'll see that the most important one is the Feynman prop. defined in Minkowski space-time.

2) G_F and G_E can be related by a Wick-rotation (analyt. cont.)

$$G_F(x,y) = \int \frac{d^3\vec{p}}{(2\pi)^3} \int_{-\infty}^{+\infty} \frac{dp^0}{2\pi} \frac{i}{(p^0)^2 - E_{\vec{p}}^2 + i0^+} e^{ip^0(y^0-x^0) - i\vec{p}(\vec{y}-\vec{x})}$$



rotate
$iy^0 = \tilde{y}^0$
$p^0 = i\tilde{p}^0$
$dp^0 = i d\tilde{p}^0$



$\vec{p} \to -\vec{p}$

$$G_E(x,y) = \int \frac{d^3\vec{p}}{(2\pi)^3} \int \frac{d\tilde{p}^0}{(2\pi)} \frac{1}{(\tilde{p}^0)^2 + \vec{E}_p^2} e^{i\tilde{p}^0(\tilde{y}^0 - \tilde{x}^0) + i\vec{p}(\vec{y}-\vec{x})}$$

3) We have seen that $G_F(x,y)$ satisfies the Klein-Gordon eq for $x \neq y$

$$[\partial_x^\mu \partial_{x\mu} + m^2]\, G_F(x,y) = \int \frac{d^4p}{(2\pi)^4} \frac{i(-p^\mu p_\mu + m^2)}{(p^0)^2 - \vec{p}^2 - m^2 + i0^+} e^{ip(y-x)}$$

$$\stackrel{!}{=} -i\delta^{(4)}(y-x)$$

4) The spectral function enjoys a relation to G_F and G_E
(→ ex. 3.2)

reading: [PS] ch 2.4