1.5. Interacting scalar fields, S-matrix

So far we have only considered free scalar fields with Hamiltonian $\hat{H}_0$.
Now we will introduce interactions $\hat{H}_{int}$, this is best dealt with in the Dirac picture:

Let $\hat{H} = \hat{H}_0 + \hat{H}_{int}$

- Schrödinger picture: $i\frac{d}{dt}|t;S\rangle = \hat{H}|t;S\rangle$, $i\frac{d}{dt}\hat{O}_S = 0$
- Heisenberg picture: $|H\rangle = |0;S\rangle$, $\hat{O}_H(t) = e^{i\hat{H}t}\hat{O}_S e^{-i\hat{H}t}$

$$\Rightarrow \quad i\frac{d}{dt}|H\rangle = 0 \qquad i\frac{d}{dt}\hat{O}_H(t) = [\hat{O}_H(t), \hat{H}]$$

- Dirac picture (I):

$$|t;I\rangle \equiv e^{i\hat{H}_0 t}|t;S\rangle$$

$$\hat{O}_I(t) \equiv e^{i\hat{H}_0 t}\hat{O}_S e^{-i\hat{H}_0 t}$$

only carries "trivial" time evolution

$$\Rightarrow \quad \boxed{i\frac{d}{dt}\hat{O}_I(t) = [\hat{O}_I(t), \hat{H}_0]}$$

$$i\frac{d}{dt}|t;I\rangle = -\hat{H}_0 e^{i\hat{H}_0 t}|t;S\rangle + e^{i\hat{H}_0 t}\hat{H}\underbrace{|t;S\rangle}_{= e^{-i\hat{H}_0 t}|t;I\rangle}$$

$$= \underbrace{e^{i\hat{H}_0 t}H_{int}e^{-i\hat{H}_0 t}}_{\hat{H}_I(t)}|t;I\rangle$$

$$\Leftrightarrow \quad \boxed{i\frac{d}{dt}|t;I\rangle = \hat{H}_I(t)\,|t;I\rangle} \qquad (*)$$

In order to solve $(*)$ we def. an op. that provides the time evolution from t_0 to time t:

$$|t;I\rangle = \hat{U}_I(t,t_0)|t_0;I\rangle$$

$$\Rightarrow \quad \boxed{\begin{aligned} i\frac{d}{dt}\hat{U}_I(t,t_0) &= \hat{H}_I(t)\,\hat{U}_I(t,t_0) \\ U_I(t_0,t_0) &= \mathbb{1} \end{aligned}} \qquad (\square)$$

b.c.

At the end one often chooses $t_0 = -\infty$ (to be able to start from free incoming states)

Eq. (II) can be rewritten as an integral eq.

$$\hat{U}_I(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} dt'\, \hat{H}_I(t')\, \hat{U}_I(t',t_0)$$

leading to an <u>iterative solution</u>:

$$\hat{U}_I(t,t_0) = \mathbb{1} - i\int_{t_0}^{t} dt_1\, \hat{H}_I(t_1) + (-i)^2 \int_{t_0}^{t} dt_1\, \hat{H}_I(t_1) \int_{t_0}^{t_1} dt_2\, \hat{H}_I(t_2) + \dots$$

$$= \sum_{n=0}^{\infty} (-i)^n \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n\, \hat{H}_I(t_1) \dots \hat{H}_I(t_n) \qquad (\Delta)$$

- recall the timeordering op T (p. 16, in $G_F(x,y)$)

$$T(\hat{H}_I(t_1) \dots \hat{H}_I(t_n)) = \begin{cases} \hat{H}_I(t_1)\, \hat{H}_I(t_2) \dots \hat{H}_I(t_n) & \text{for } t_1 > t_2 > \dots > t_n \\ \hat{H}_I(t_2)\, \hat{H}_I(t_1)\, \hat{H}_I(t_3) \dots \hat{H}_I(t_n) & \text{for } t_2 > t_1 > t_3 > \dots > t_n \\ \text{etc} & \end{cases}$$

- the product inside the integrand is time ordered
- there are $n!$ possibilities to order n operators

$\Rightarrow$ if we integrate the timeordered product over <u>all times</u> this will give $n!$ identical contributions equal to the ~~there~~ in (Δ):

$$\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 \dots \int_{t_0}^{t} dt_n\, T\{\hat{H}_I(t_1)\, \hat{H}_I(t_2) \dots \hat{H}_I(t_n)\}$$

$$= n! \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n\, \hat{H}_I(t_1)\, \hat{H}_I(t_2) \dots \hat{H}_I(t_n)$$

(this identity is also true for commuting objects $\hat{H}(t_i) \to f(t_i)$)

$$\Rightarrow \quad \hat{U}_I(t,t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^{t} dt_1 \dots \int_{t_0}^{t} dt_n\, T\{H_I(t_1) \dots H_I(t_n)\}$$

$$\Leftrightarrow \quad \boxed{\hat{U}_I(t,t_0) = T\{\exp[-i\int_{t_0}^{t} dt'\, H_I(t')]\}}$$

To summarise, we have found a formal solution for the general time evolution. For this we only need the operator

$$\hat{H}_I(t) = e^{i\hat{H}_0 t}\hat{H}_{int}e^{-i\hat{H}_0 t}$$

If $\hat{H}_{int}$ is a polynomial in $\hat{\phi}_S$, e.g. $H_{int} = \lambda\hat{\phi}_S^4$, it holds

$$\hat{H}_I(t) = e^{i\hat{H}_0 t}\sum_n c_n \underbrace{\hat{\phi}_S \cdots \hat{\phi}_S}_{n \text{ times}} e^{-i\hat{H}_0 t}, \quad H_{int} = \sum_n^{\deg H_{int}} c_n (\hat{\phi}_S)^n$$

insert $e^{-i\hat{H}_0 t} e^{i\hat{H}_0 t}$ at $\wedge \cdots \wedge$

$$= \sum_n c_n \underbrace{\hat{\phi}_I(t) \cdots \hat{\phi}_I(t)}_{n \text{ times}}$$

- the fields $\hat{\phi}_I(t)$ are the Heisenberg field of the <u>non-interacting</u> system which we have studied before. Thus we can express the interaction $\hat{H}_I(t)$ in terms of free fields.

The $\hat{\phi}_I(t)$ are often called in-fields (ingoing) $\hat{\phi}_{in}(t)$

Def <u>S-matrix</u> (scattering matrix)

$$\hat{S} \equiv \hat{U}_I(\infty, -\infty) = T\left\{\exp\left[-i\int_{-\infty}^{\infty} dt' \hat{H}_I(t')\right]\right\}$$

It holds that

$$\hat{H}_{int}^\dagger = \hat{H}_{int}, \quad \hat{H}_0^\dagger = \hat{H}_0 \Rightarrow \hat{H}_I(t)^\dagger = \hat{H}_I(t)$$

and thus

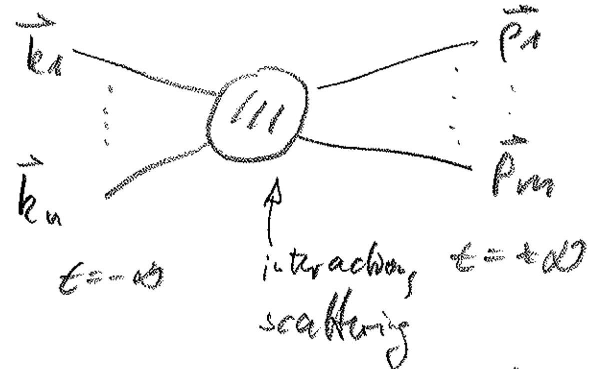
$$\boxed{\hat{S}^\dagger = T\left\{\exp\left[+i\int_{-\infty}^{\infty} dt' H_I(t')\right]\right\}}$$

note $T\{(AB)^\dagger\} = T\{B^\dagger A^\dagger\} = T\{A^\dagger B^\dagger\}$ as T determines the order

$$\boxed{\hat{S}^\dagger\hat{S} = \hat{S}\hat{S}^\dagger = \mathbb{1}} \rightarrow \text{ex } 3.4$$

The S-matrix is unitary

We still need to define physical states that are meaningful.
Here we shall follow the conventions from particle physics, where initial and final state will be free particles (on shell):



• The in-states will be chosen as $\hat{H}_0$ eigenstates (see p. 14)

$$|\vec{k}_1, \dots, \vec{k}_n; in\rangle = \hat{a}^\dagger_{\vec{k}_1} \dots \hat{a}^\dagger_{\vec{k}_n} |0\rangle$$

• time evolution brings us to

$$\hat{S}\,|\vec{k}_1 \dots \vec{k}_n; in\rangle$$

• at the end we measure the outgoing free $\hat{H}$ eigenstates:

$$S_{fi} \equiv \underbrace{\langle \vec{p}_1, \dots, \vec{p}_m; in|\,\hat{S}}|\vec{k}_1 \dots \vec{k}_n; in\rangle$$

final initial $\equiv \langle \vec{p}_1, \dots \vec{p}_m; out|\vec{k}_1 \dots \vec{k}_n; in\rangle$

so $\quad |\vec{p}_1, \dots, \vec{p}_m; out\rangle = \hat{S}^\dagger |\vec{p}_1 \dots \vec{p}_m; in\rangle$

We call S_{fi} a <u>scattering matrix element</u>. As in particle phys.
$|S_{fi}|$ determines decay rates, scattering cross sections etc.