1.6 Green functions

In the previous chapter we have defined the scattering matrix element $S_{fi} = \langle \vec{p}_1, \dots, \vec{p}_m; in | \hat{S} | \vec{k}_1, \dots, \vec{k}_n; in \rangle$, in which all can be expressed in terms of free fields $\hat{\phi}_{in}(t) = \hat{\phi}_I(t)$. If we were to expand $\hat{S}$ into a perturbative series in $\hat{\phi}_I(t)$ we could already now compute physical quantities. But it is useful, prior to that, to relate S_{fi} to "nice" objects, the n-point Green's functions:

- general def (see also p. 16, G_F)

$$G_T^{(n)}(x_1, \dots, x_n) \equiv \langle 0 | T\{\hat{\phi}_H(x_1) \dots \hat{\phi}_H(x_n)\} | 0 \rangle$$

, $\hat{\phi}_H$ Heisenberg picture

- In case $G_T^{(n)}$ contains partly factorised terms, such as $\langle 0 | T\{\ \} | 0 \rangle \langle 0 | T\{\ \} | 0 \rangle$ we call these parts disconnected. The n-point fnct. containing only connected (c) parts is def as

$$G_{T,c}^{(n)}(x_1, \dots, x_n) \equiv \langle 0 | T\{\hat{\phi}_H(x_1) \dots \hat{\phi}_H(x_n)\} | 0 \rangle_c$$

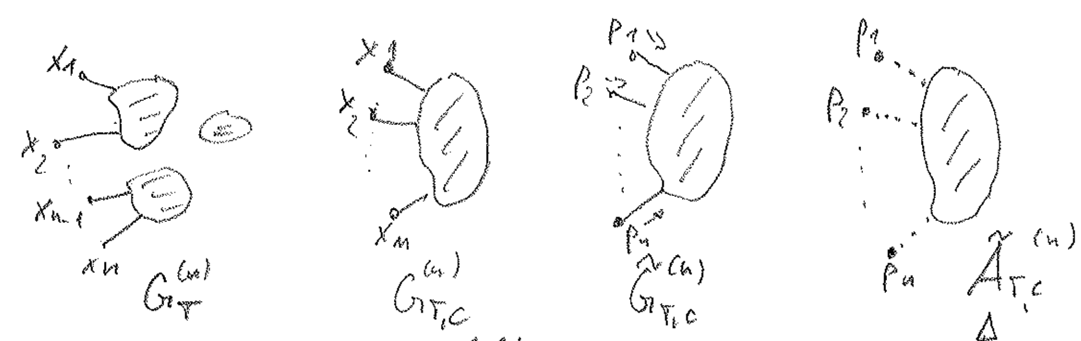
- its multi-dim. Fourier trafo is def as

$$(2\pi)^4 \delta^{(4)}(p_1 + \dots + p_n)\, \tilde{G}_{T,c}^{(n)}(p_1, \dots, p_n) \equiv \int d^4x_1 \dots \int d^4x_n\, G_{T,c}^{(n)}(x_1, \dots, x_n)\, e^{i(p_1 \cdot x_1 + \dots + p_n \cdot x_n)}$$

- amputation of the external legs (momenta)

$$\tilde{A}_{T,c}^{(n)}(p_1, \dots, p_n) \equiv \left[\tilde{G}_{T,c}^{(2)}(p_1, -p_1)\right]^{-1} \dots \left[\tilde{G}_{T,c}^{(2)}(p_n, -p_n)\right]^{-1} \tilde{G}_{T,c}^{(n)}(p_1, \dots, p_n)$$

graphically



these will appear in the transition matrix of $\hat{S}$

- Note: already the full 2-point fnct. $\tilde{G}_{T,c}^{(2)}$ = (diagram) is very nontrivial and contains most info about the interactions

Examples for connected correlation function

1-point

$$G_T^{(1)}(x_1) = \langle 0|\hat{T}\hat{\phi}_H(x_1)|0\rangle = 0 \text{ is connected (and zero)}$$
$$= G_{T,c}^{(1)}(x_1)$$

2-point

$$G_{T,c}^{(2)}(x_1,x_2) \equiv \langle 0|T\{\hat{\phi}_H(x_1)\,\hat{\phi}_H(x_2)\}|0\rangle - \underbrace{\langle 0|\hat{\phi}_H(x_1)|0\rangle}_{0}\underbrace{\langle 0|\hat{\phi}_H(x_2)|0\rangle}_{0}$$
$$= G_T^{(2)}(x_1,x_2)$$

denote $G_{T(c)}^{(n)}(x_1 \ldots x_n) = \langle 1\,2 \ldots n\rangle_{(c)}$

3 Point:

$$G_{T,c}^{(3)}(x_1 x_2, x_3) = \langle 1\,2\,3\rangle_c = \langle 1\,2\,3\rangle - \langle 1\,2\rangle\langle 3\rangle - \langle 1\,3\rangle\langle 2\rangle - \langle 2\,3\rangle\langle 1\rangle$$
$$= \langle 1\,2\,3\rangle$$

is connected (and 0 as we will see)

4-points: $\langle 1234\rangle \neq \langle 1234\rangle_c$ (check)

we will later formalise the computation of $G_T^{(n)}$ using Wick's theorem, and of $G_{T,c}^{(n)}$ by defining a generating functional

Let's express the n-point Greenfct. in terms of fields $\hat{\phi}_I$ in the Dirac pic.

- the relation of $\hat{\phi}_I$ and $\hat{\phi}_H$:
  - $|t;I\rangle = e^{i\hat{H}_0 t}|t;S\rangle = e^{i\hat{H}_0 t}e^{-i\hat{H}t}|0;S\rangle = \underbrace{e^{i\hat{H}_0 t - i\hat{H}(t-t_0)}e^{-i\hat{H}_0 t_0}}_{\hat{=}\ \hat{U}_I(t,t_0)}|t_0;I\rangle$ $\Rightarrow$ $\hat{=}\ \hat{U}_I(t,t_0)$ (→ eq 3.3)
  - $\hat{\phi}_H(t) = e^{i\hat{H}t}\hat{\phi}_S e^{-i\hat{H}t}$
    $= e^{i\hat{H}t}e^{-i\hat{H}_0 t}\hat{\phi}_I(t)e^{i\hat{H}_0 t}e^{-i\hat{H}t} = \hat{U}_I^\dagger(t,0)\,\hat{\phi}_I(t)\,\hat{U}_I(t,0)$

- Consider $G_T^{(n)}$, and let $t > t_1 > t_2 > \ldots > t_n$:

$$G_T^{(n)} = \langle 0|\hat{\phi}_H(x_1)\ldots\hat{\phi}_H(x_n)|0\rangle$$
$$= \langle 0|\hat{U}_I^\dagger(t_1,0)\hat{\phi}_I(x_1)\hat{U}_I(t_1,0)\hat{U}_I^\dagger(t_2,0)\hat{\phi}_I(x_2)\hat{U}_I(t_2,0)\ldots\hat{U}_I^\dagger(t_n,0)\hat{\phi}_I(x_n)\hat{U}_I(t_n,0)|0\rangle$$

$\hat{1\!\!1} = \hat{U}_I^\dagger(t,0)\hat{U}_I(t,0)$ $\qquad$ $\hat{1\!\!1} = \hat{U}_I^\dagger(-t,0)\hat{U}_I(-t,0)$

$$\overset{\text{eq 3.3}}{=} \langle 0|\hat{U}_I^\dagger(t,0)\{\hat{U}_I(t,t_1)\hat{\phi}_I(x_1)\hat{U}_I(t_1,t_2)\ldots\ldots\phi_I(x_n)U_I(t_n,-t)\}\hat{U}_I(-t,0)|0\rangle$$

$$\hat{=}\ T\{\hat{\phi}_I(x_1)\ldots\hat{\phi}_I(x_n)\underbrace{\hat{U}_I(t,t_1)\ldots U_I(t_n,-t)}_{\hat{=}\ \hat{U}_I(t,-t)}\}$$

$\hat{1\!\!1} = \sum_m |m\rangle\langle m|$ $\qquad$ $\hat{1\!\!1} = \sum_m |m\rangle\langle m|$

- Letting $t \to +\infty$ we can assume that the ground state $|0\rangle$ dominates the time evolution (add a small Im-part in t)

$$\Rightarrow \langle 0|\hat{U}_I(-t,0)|0\rangle \overset{t\gg 1}{=} e^{i\alpha t}, \quad \langle 0|\hat{U}_I^\dagger(t,0)|0\rangle \overset{\text{eq 3.3}}{=} \langle 0|U_I(0,t)|0\rangle \overset{t\gg}{=} e^{i\beta t}$$

$$\Rightarrow \langle 0|\hat{U}_I(+t,0)|0\rangle = e^{-i\alpha t}, \quad \langle 0|\hat{U}_I(0,-t)|0\rangle = e^{-i\beta t}$$

- $\Rightarrow$ factors above

$$\langle 0|U_I^\dagger(t,0)|0\rangle\langle 0|\hat{U}_I(-t,0)|0\rangle = e^{i\alpha t}e^{i\beta t} = \frac{1}{e^{-i\alpha t}e^{-i\beta t}} \overset{t\gg 1}{=} \frac{1}{\underbrace{\langle 0|\hat{U}_I(t,0)\hat{U}_I(0,-t)|0\rangle}_{\hat{=}\ \hat{U}_I(t,-t)}}$$

($1\!\!1 = \sum_m |m\rangle\langle m|$)

$$\overset{\lim}{\underset{t\to\infty}{\Rightarrow}} \quad \boxed{G_T^{(n)} = \frac{\langle 0|T\{\hat{\phi}_I(x_1)\ldots\hat{\phi}_I(x_n)\hat{S}\}|0\rangle}{\langle 0|S|0\rangle}}$$

as $\lim_{t\to\infty}\hat{U}_I(t,-t) = \hat{S}$

Examples for matrix elements for the free scalar field:

p (NL) $\hat{\phi}(x) = \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2E_{\vec{p}}}} \left[\hat{a}_{\vec{p}} e^{-ipx} + \hat{a}^\dagger_{\vec{p}} e^{ipx} \right]$

$H = H_0$, it holds $i\partial_0 \hat{\phi}(x^0,\vec{x}) = [\hat{\phi}(x^0,\vec{x}), \hat{H}_0]$

so here $\hat{\phi}_H(x) = \hat{\phi}_I(x) = e^{i\hat{H}_0 t} \hat{\phi}_S e^{-i\hat{H}_0 t}$

$$\Rightarrow \langle 0 | \hat{\phi}_I(x) | \vec{k}, in \rangle = \langle 0 | \hat{\phi}_I(x) \hat{a}^\dagger_k | 0 \rangle = \frac{1}{\sqrt{(2\pi)^3 2E_{\vec{p}}}} e^{-ik\cdot x}$$

$$\langle \vec{p}; in | \hat{\phi}_I(x) | 0 \rangle = \langle 0 | \hat{a}_{\vec{p}} \hat{\phi}_I(x) | 0 \rangle = \frac{1}{\sqrt{(2\pi)^3 2E_{\vec{p}}}} e^{ip\cdot x}$$

- <u>note</u>: there are many different conventions in the literature regarding the normalisation $\frac{1}{\sqrt{(2\pi)^3 2E_{\vec{p}}}}$, $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$. Only the final answers are the same.

<u>S-matrix elements and LSZ-reduction</u> [Lehmann, Symanzik, Zimmermann 1955]

we can write

$$S_{fi} = \langle \vec{p}_1, \ldots, \vec{p}_m; in | \hat{S} | \vec{k}_1 \ldots \vec{k}_n, in \rangle =$$

$$= \delta_{mn} \langle \vec{p}_1, \ldots, \vec{p}_m; in | \vec{k}_1 \ldots \vec{k}_n, in \rangle + (2\pi)^4 i\, \delta^{(4)}(\vec{p}_1 + \ldots + \vec{p}_m - \vec{k}_1 - \ldots - \vec{k}_n)\, T(\vec{p}_1, \ldots, \vec{k}_n)$$

The transition matrix elements T can be written as a product of amplitudes $\mathcal{M}$ that are in turn proportional to the amputated Green's funct., $\mathcal{M} \sim \tilde{A}^{un}_{T.C}$. This is called LSZ reduction

(for more see [IZ] ch 5-1-2 – 5-1-4 pp. 202–212]