1.7 Perturbation Theory

We have learned that all interesting info is contained in the n-point Green fuct.

$$G_T^{(n)}(x_1,\dots,x_n) = \frac{\langle 0|T\{\hat\phi_I(x_1)\dots\hat\phi_I(x_n)\hat S\}|0\rangle}{\langle 0|\hat S|0\rangle}$$

with

$$\hat S = T\left\{\exp\left[-i\int_{-\infty}^{\infty}dt\,\hat H_I(t)\right]\right\}$$

and its Fourier-trafo. We will now determine these in some approximation scheme: if $\hat H_I$ is in some sense small, i.e. it contains a small parameter, we can treat H_I as a perturbation of the free field wit $\hat H_0$.

For illustration consider the following example

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{m^2}{2}\phi^2 + \mathcal{L}_{int}$$

$$\mathcal{L}_{int} = -\frac{1}{4!}\lambda\phi^4 \qquad \phi^4 \text{ scalar field th.}$$

Then $\hat H_I = \int d^3x\, \mathcal{H}_{int}\Big|_{\phi\to\phi_I} = \int d^3x\, \frac{\lambda}{4!}\hat\phi_I^4(x)$

$(= e^{i\hat H_0 t}\hat H_{int}e^{-i\hat H_0 t})$

The operator $\hat S$ will be expanded into a Taylor series, according to its def. above. Because of

$$T\{\hat\phi_I(x_1)\dots\hat\phi_I(x_n)\,T\{\hat\phi_I(y_1)\dots\hat\phi_I(y_m)\}\} = T\{\hat\phi_I(x_1)\dots\hat\phi_I(x_n)\hat\phi_I(y_1)\dots\hat\phi_I(y_m)\}$$

we have

$$\boxed{G_T^{(n)} = \frac{\sum_{j=0}^{\infty}\left(\frac{-i\lambda}{4!}\right)^j\frac{1}{j!}\langle 0|T\{\hat\phi_I(x_1)\dots\hat\phi_I(x_n)\left[\int d^4y\,\hat\phi_I^4(y)\right]^j\}|0\rangle}{\sum_{j=0}^{\infty}\left(\frac{-i\lambda}{4!}\right)^j\frac{1}{j!}\langle 0|T\{\left[\int d^4y\,\hat\phi^4(y)\right]^j\}|0\rangle}}$$

In order to compute these objects we use Wick's Theorem

[note: $T\{\hat\phi_I(x_1)\hat\phi_I(x_2)\}\,T\{\hat\phi_I(x_3)\hat\phi_I(x_4)\} \neq T\{\hat\phi_I(x_1)\dots\hat\phi_I(x_4)\}$]

Wick's Theorem:

$= \hat{\phi}_{H_0} = \phi(x)$ free field!

we have $G_F(x,y) = \langle 0 | T\{\hat{\phi}_I(x)\,\hat{\phi}_I(y)\} | 0 \rangle = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{ip(x-y)}}{p^2 - m^2 + i0^+}$

and from normal ordering $0 = \langle 0 | : \hat{\mathcal{O}} : | 0 \rangle$ for any op. $\mathcal{O}$

- in general: $\hat{\phi}_I(x_1)\,\hat{\phi}(x_2) = :\hat{\phi}_I(x_1)\hat{\phi}_I(x_2): + \langle 0 | \hat{\phi}_I(x_1)\,\hat{\phi}_I(x_2) | 0 \rangle$
- we will now reduce all n-point $G^{(n)}$ to these objects:

ex 4.2 gives $:\hat{\phi}_I(x_1)\,\hat{\phi}_I(x_2): = :\hat{\phi}_I(x_2)\,\hat{\phi}_I(x_1):$

Thus
$$\begin{aligned}
T\{\phi_I(x)\,\phi_I(y)\} &= \phi_I(x)\phi_I(y)\,\Theta(x^0-y^0) + \hat{\phi}_I(y)\hat{\phi}_I(x)\,\Theta(y^0-x^0) \\
&= :\phi_I(x)\phi_I(y):\,[\Theta(x^0-y^0)+\Theta(y^0-x^0)] + \langle 0 | T\{\hat{\phi}_I(x)\hat{\phi}_I(y)\} | 0 \rangle \\
&= T\{:\phi_I(x)\phi_I(y):\} + G_F(x,y) \\
&\overset{ex\,4.2}{=} :\phi_I(x)\,\phi_I(y): + \underbrace{\hat{\phi}_I(x)\,\hat{\phi}(y)}
\end{aligned}$$

($G_F(x,y) \equiv$ contraction) def contractions

Wick's Theorem:

$$\begin{aligned}
T\{\hat{\phi}_I(x_1)\ldots\hat{\phi}_I(x_n)\} &= :\hat{\phi}_I(x_1)\ldots\hat{\phi}_I(x_n): \\
\text{1 contraction} \quad &+ \sum_{j,k} :\hat{\phi}_I(x_1)\ldots\cancel{\hat{\phi}_I(x_j)}\ldots\cancel{\hat{\phi}(x_k)}\ldots\hat{\phi}_I(x_n):\ \underbrace{\hat{\phi}_I(x_j)\hat{\phi}(x_k)} \\
\text{2 —"—} \quad &+ \sum_{j,k,l,m} :\hat{\phi}_I(x_1)\ldots\hat{\phi}_I(x_n):\ \underbrace{\hat{\phi}_I(x_j)\hat{\phi}_I(x_k)}\ \underbrace{\hat{\phi}_I(x_l)\hat{\phi}_I(x_m)} \\
&\qquad \text{no } x_{j,k,l,m} \\
\vdots \quad & \qquad \vdots \\
\text{all —"—} \quad &+ \sum_{j_1\ldots j_n} \underbrace{\hat{\phi}_I(x_{j_1})\hat{\phi}_I(x_{j_2})}\ldots \begin{cases} \underbrace{\hat{\phi}_I(x_{j_{n-1}})\hat{\phi}_I(x_{j_n})} & n \text{ even} \\ \hat{\phi}(x_{j_n}) & n \text{ odd} \end{cases}
\end{aligned}$$

ex 4.3: check for $n = 3$, general proof later

If we use this result all terms with $:\,..\,:$ in $\langle 0|T\{..\}|0\rangle$ will give zero. In particular we get

$$n \text{ even}: \quad \langle 0|T\{\hat\phi_I(x_1)\dots\hat\phi_I(x_n)\}|0\rangle = \sum_{\substack{\text{all}\\ \text{possibilities}}} \prod_{j=1}^{n/2} \underbrace{\hat\phi_I(x_{2j-1})\,\hat\phi_I(x_{2j})}$$

$$n \text{ odd}: \quad \langle 0|T\{\hat\phi_I(x_1)\dots\hat\phi_I(x_n)\}|0\rangle = 0$$

To illustrate this result let us consider our example ϕ^4-theory and expand to $O(\lambda^0)$, $O(\lambda^1)$. We use the notation

$$\underbrace{\hat\phi_I(x)\,\hat\phi_I(y)} \;\hat{=}\; x \circ\!\!-\!\!-\!\!\circ\, y$$

- example 4 point function $O(\lambda^0)$:

$$G_T^{(4)}(x_1,x_2,x_3,x_4) = \frac{\langle 0|T\{\hat\phi_I(x_1)\hat\phi_I(x_2)\hat\phi_I(x_3)\hat\phi_I(x_4)\}|0\rangle}{\underbrace{\langle 0|0\rangle}_{=1}}$$

$$\overset{\text{wick}}{=} \underbrace{\hat\phi_I(x_1)\,\hat\phi_I(x_2)}\;\underbrace{\hat\phi_I(x_3)\,\hat\phi_I(x_4)}$$

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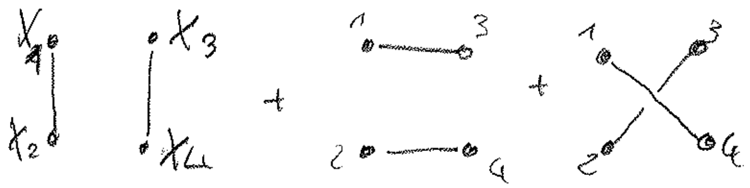
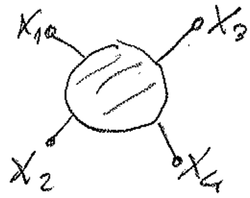
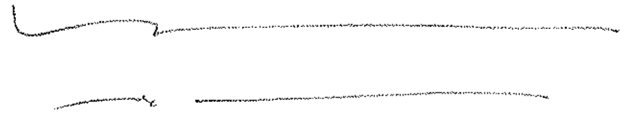
$+$

$=$ 

$\Rightarrow$ the full 4-point funct. is

$G_T^{(4)}$: 

$=$  $+\,O(\lambda)$

Note : the order $O(\lambda^0)$ does not contain connected terms

(3-1)

expand $G_\gamma^{(4)}$ to order $\mathcal{O}(\lambda^1)$:

$$G_\gamma^{(4)}(x_1,x_2,x_3,x_4) = \frac{\langle 0|T\{\hat\phi_I(x_1)\hat\phi_I(x_2)\hat\phi_I(x_3)\hat\phi_I(x_4)\left[1-\left(\frac{i\lambda}{4!}\right)\int d^4y\, \hat\phi_I(y)^4\right]\}|0\rangle}{\langle 0|T\{1-\frac{i\lambda}{4!}\int d^4y\, \phi_I(y)^4\}|0\rangle}$$

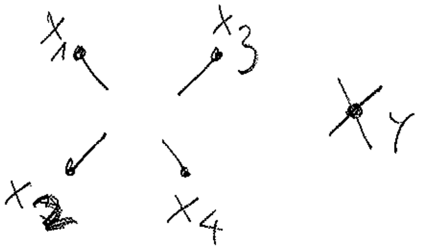
$$= \mathcal{O}(\lambda^0) - \frac{i\lambda}{4!}\int d^4y \Big[\langle 0|T\{\hat\phi(x_1)\hat\phi(x_2)\hat\phi(x_3)\hat\phi(x_4)\hat\phi_I(y)\hat\phi_I(y)\hat\phi_I(y)\hat\phi_I(y)\}|0\rangle$$

$$[\dots] := \quad - \langle 0|T\{\hat\phi_I(x_1)\hat\phi_I(x_2)\hat\phi_I(x_3)\hat\phi_I(x_4)\}|0\rangle\langle 0|T\{\hat\phi_I(y)\hat\phi_I(y)\hat\phi_I(y)\hat\phi_I(y)\}|0\rangle\Big]$$

$\mathcal{O}(\lambda^0)$-term

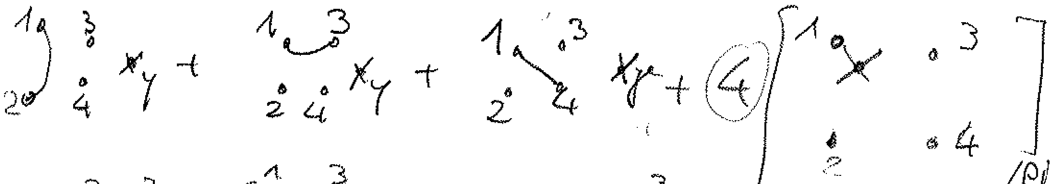
graphical problem:

compute all contractions in first part:

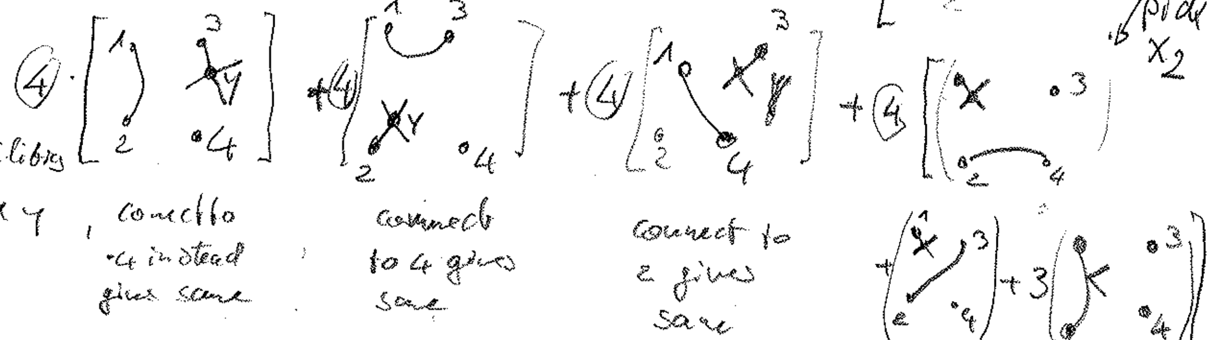


- we need to contract at least $1\,x_i$ with 1_y, the other part is subtracted

pick x_1, do 1 contr.: $[\dots] =$



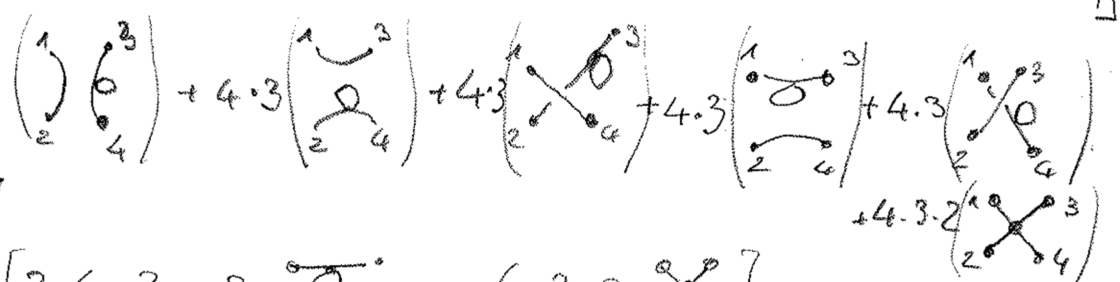
2. contr. $=$ 4 · [...] + 4 [...] + 4 [...] + 4 [...] + [...] + 3 [...]



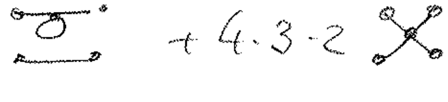
4 possibilities to pick y, connect to 4 instead gives same; connect to 4 gives same; connect to 2 gives same; pick x_2

3. contr. $[\dots] = 4\cdot 3\,(\dots) + 4\cdot 3\,(\dots) + 4\cdot 3\,(\dots) + 4\cdot 3\,(\dots) + 4\cdot 3\,(\dots) + 4\cdot 3\cdot 2\,(\dots)$

3 poss. for picking y



$\Rightarrow \quad [\dots] = \left[2\cdot 4\cdot 3\cdot 3 \ (\dots) + 4\cdot 3\cdot 2\ (\dots)\right] = 96$



[check: pick x_1 $\Rightarrow$ $\exists$ $7\cdot 5\cdot 3\cdot 1$ possibilities, minus $3\cdot 3$ (x_i with x_i & y with y only)

$= 7\cdot 5\cdot 3 - 3\cdot 3 = 105 - 9 = 96$ ✓]