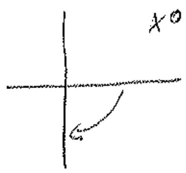
2. Pathintegral quantisation of scalar fields

The idea is to replace the Green's function as a vacuum exp. value of the field operators and the S-matrix operator by a functional (path) integral over fields and the Hamilton function which are no longer operator valued. We shall explain the idea first in the quantum mechanical setting

2.1. Path integral in QM : 1D

Preparation i) We consider all quantities in Euclidean space, i.e. after Wick rotation. Continuation to Minkowski is done last



$x^0 = -i\tilde{x}^0 \iff t = -i\tau$, scalar product $-x\cdot x = \tilde{x}^{0\,2} + \vec{x}^2$

Heisenberg picture : $\hat{x}_H = e^{i\hat{H}t}\hat{x}e^{-i\hat{H}t} = e^{\hat{H}\tau}\hat{x}e^{-\hat{H}\tau}$

(for more details & example HO recall Lecture 2 $\;x \to t,\ \phi \to x(t)$)

ii) Euclidean n-point function $G_E^{(n)}(\tau_1,\dots,\tau_n) = \langle 0|T\{\hat{x}_H(\tau_1)\dots\hat{x}_H(\tau_n)\}|0\rangle$

Consider $\beta > \tau_1,\dots,\tau_n \geq 0$

def $$\boxed{G_\beta^{(n)}(\tau_1,\dots,\tau_n) \equiv \frac{\mathrm{Tr}\left[e^{-\beta\hat{H}}\,T\{\hat{x}_H(\tau_1)\dots\hat{x}_H(\tau_n)\}\right]}{\mathrm{Tr}\left[e^{-\beta\hat{H}}\right]}} \xrightarrow[\beta\to\infty]{ex\ 5} G_E^{(n)}$$

iii) Notation : $\mathrm{Tr}[O] = \begin{cases} \int dp\, \langle p|O|p\rangle = \int dx\, \langle x|O|x\rangle & \text{cont.} \\ \quad\text{momentum space} \qquad \text{coord. space} & \\ \sum_m \langle m|O|m\rangle & \text{discrete} \end{cases}$

$\sum_m |m\rangle\langle m|$ or $\boxed{\int dx\, |x\rangle\langle x| = \mathbb{1} = \int \frac{dp}{B}|p\rangle\langle p|}$ completeness relation

• $\langle p|\hat{p}|x\rangle = p\langle p|x\rangle = -i\partial_x\langle p|x\rangle \implies \boxed{\langle p|x\rangle = A e^{ipx}}$

determine A,B : $\mathbb{1} = \int dx \int\frac{dp}{B}\int\frac{dp'}{B}\,|p\rangle \underbrace{\langle p|x\rangle\langle x|p'\rangle}_{|A|^2 e^{ix(p-p')}}\langle p'| = \frac{|A|^2}{B^2}\int dp\int dp'\,\delta(p-p')\,2\pi\,|p\rangle\langle p'|$

$\implies \frac{|A|^2}{B}2\pi = 1$ choose $A = 1,\ B = 2\pi$ (with $\hbar = 1$)

Fix time order $\tau_1 > \tau_2 > \ldots > \tau_n \geq 0$

$$\mathrm{Tr}\left[e^{-\beta \hat{H}} \hat{X}_H(\tau_1) \ldots \hat{X}_H(\tau_n)\right] = \int dx \langle x | \underbrace{e^{-\beta \hat{H}} e^{\hat{H}\tau_1}}_{e^{-(\beta-\tau_1)\hat{H}}} \hat{X} \underbrace{e^{-\hat{H}\tau_1} e^{\hat{H}\tau_2}}_{e^{-(\tau_1-\tau_2)\hat{H}}} \hat{X} \ldots e^{\hat{H}\tau_n} \hat{X} e^{-\hat{H}\tau_n} | x \rangle$$

- define $\varepsilon = \frac{\beta}{N}$, $N \gg 1$
- split each factor

$$e^{-(\beta-\tau_1)\hat{H}} = \left(e^{-\varepsilon \hat{H}}\right)^{\frac{\beta-\tau_1}{\varepsilon}} = \overbrace{e^{-\varepsilon \hat{H}} e^{-\varepsilon \hat{H}} \ldots e^{-\varepsilon \hat{H}}}^{\frac{\beta-\tau_1}{\varepsilon} \text{ times}}$$

$$e^{-(\tau_1-\tau_2)\hat{H}} = \left(e^{-\varepsilon \hat{H}}\right)^{\frac{\tau_1-\tau_2}{\varepsilon}} = e^{-\varepsilon \hat{H}} \ldots$$

- total number of times $\sum = \frac{\beta-\tau_1}{\varepsilon} + \frac{\tau_1-\tau_2}{\varepsilon} + \ldots + \frac{\tau_n}{\varepsilon} = \frac{\beta}{\varepsilon} = N$

- we insert an identity $\mathbb{1} = \int dx_{i+1} \int \frac{dp_i}{2\pi} |x_{i+1}\rangle\langle x_{i+1}|p_i\rangle\langle p_i|$

before each factor $e^{-\varepsilon \hat{H}}$: $\underset{\mathbb{1}}{\wedge} e^{-\varepsilon \hat{H}} \underset{\mathbb{1}}{\wedge} e^{-\varepsilon \hat{H}} \ldots$ or $\underset{\mathbb{1}}{\wedge}\left(e^{-\varepsilon \hat{H}} \hat{X}\right)\underset{\mathbb{1}}{\wedge} e^{-\varepsilon \hat{H}}$

and rename $|x\rangle = |x_1\rangle$

(in total N identities $\mathbb{1}$)

$$\Rightarrow \mathrm{Tr}[\;] = \left(\prod_{i=1}^{N} \int dx_{i+1} \int \frac{dp_i}{2\pi}\right) \int dx_1 \left[\overbrace{\langle x_1 | x_{N+1}\rangle}^{\delta_{1,N+1}} \langle x_{N+1} | p_N \rangle \underline{\langle p_N | e^{-\varepsilon \hat{H}} | x_N \rangle} \langle x_N | \ldots\right.$$

$$\ldots e^{-\varepsilon \hat{H}} |x_j\rangle\langle x_j | p_{j-1}\rangle \underline{\langle p_{j-1} | e^{-\varepsilon \hat{H}} \hat{X} | x_{j-1}\rangle} \ldots |x_2\rangle\langle x_2|p_1\rangle\langle p_1| e^{-\varepsilon \hat{H}} |x_1\rangle$$

- we have $\langle p_j | \hat{p} = p_j \langle p_j |$, $\hat{X}|x_j\rangle = x_j |x_j\rangle$ — No operator!

if $\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$ $\Rightarrow$ $\langle p_j | \hat{H} | x_j \rangle = \left(\frac{p_j^2}{2m} + V(x_j)\right) \langle p_j | x_j \rangle = H(p_j, x_j) \langle p_j | x_j \rangle$

(or any $f(\hat{p}) + h(\hat{x})$)

<u>But</u> $\langle p_j | \hat{H}^2 | x_j \rangle \neq H(p_j, x_j)^2 \langle p_j | x_j \rangle$

$\Rightarrow$ expand $e^{-\varepsilon \hat{H}} = \mathbb{1} - \varepsilon \hat{H} + O(\varepsilon^2)$

$$\Rightarrow \underline{\langle p_j | e^{-\varepsilon \hat{H}} | x_j \rangle} = \left(1 - \varepsilon H(p_j, x_j) + O(\varepsilon^2)\right) \langle p_j | x_j \rangle$$
$$= \underline{\exp\left[-\varepsilon H(p_j, x_j) + O(\varepsilon^2) + i p_j x_j\right]}$$

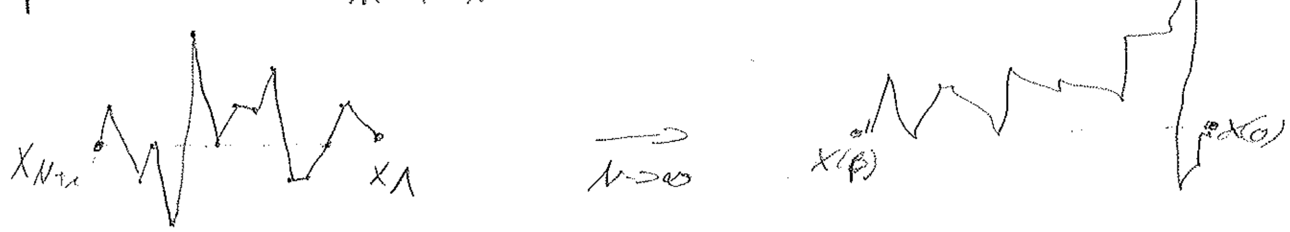
ditto $\underline{\langle p_j | e^{-\varepsilon \hat{H}} \hat{X} | x_j \rangle} = \underline{\exp\left[-\varepsilon H(p_j, x_j) + O(\varepsilon^2) + i p_j x_j\right] x_j}$

Collecting all factors we have

$$\mathrm{Tr}\left[e^{-\beta \hat H}\hat x_H(\tau_1)\ldots\hat x_H(\tau_n)\right] = \prod_{j=1}^{N}\int dx_j\int\frac{dp_j}{2\pi}\exp\Big[-\overbrace{\sum_{\ell=1}^{N}\varepsilon}^{\int d\tau}\Big[H(p_\ell,x_\ell)+ip_\ell\frac{(x_{\ell+1}-x_\ell)}{\varepsilon}+O(\varepsilon)\Big]\Big]$$

$$\cdot\; x_{j_1}\cdots\, x_{j_n}\qquad j_1 = N-\frac{\beta-\tau_1}{\varepsilon} = \frac{\tau_1}{\varepsilon},\quad j_n=\frac{\tau_n}{\varepsilon}+1\qquad \varepsilon=\frac{\beta}{N}\qquad x_{N+1}=x_1$$

L.O.: product of class actions · phase of particles at $x_1 \ldots x_N$



path $\neq$ path of least action (classical particle)

limit $N\to\infty$ defines path integral (functional integral)

$$\Rightarrow\boxed{\mathrm{Tr}\left[e^{-\beta\hat H}\hat x_H(\tau_1)\ldots\hat x_H(\tau_n)\right]=\int\limits_{x(\beta)=x(0)}\mathcal{D}x\int\frac{\mathcal{D}p}{2\pi}\exp\left[-\int_0^\beta d\tau\left(H(p(\tau),x(\tau))+ip(\tau)\dot x(\tau)\right)\right]\cdot x(\tau_1)\cdot\ldots\cdot x(\tau_n)}$$

ditto $\mathrm{Tr}[e^{-\beta\hat H}]$ without insertions

- if we use explicitly $\underline{H(p,x)=\frac{p^2}{2m}+V(x)}$ we can int. out the p_j's

$$\int_{-\infty}^{\infty}\frac{dp_j}{2\pi}\exp\left[-\varepsilon\left(\frac{p_j^2}{2m}+ip_j\frac{(x_{j+1}-x_j)}{\varepsilon}\right)\right]=\frac{1}{2\pi}\sqrt{\frac{2m\pi}{\varepsilon}}\exp\left[-\frac{(x_{j+1}-x_j)^2}{4\cdot\frac{\varepsilon}{2m}}\right]$$

as $$\int_{-\infty}^{\infty}dp\, e^{-ap^2-ibp}=\int_{-\infty}^{\infty}dp\, e^{-a(p+\frac{ib}{2a})^2}\cdot e^{-\frac{b^2}{4a}}=\sqrt{\frac{\pi}{a}}\,e^{-\frac{b^2}{4a}}\quad \text{Gauß}$$

shift + Cauchy

$$\Rightarrow \mathrm{Tr}\left[e^{-\beta\hat H}\right]=\lim_{N\to\infty}\int\frac{\prod^N dx_j}{\sqrt{\frac{2\pi\varepsilon}{m}}}\exp\left[-\sum_{j=1}^N\varepsilon\left[\frac{m}{2}\left(\frac{x_{j+1}-x_j}{\varepsilon}\right)^2+V(x_j)+O(\varepsilon)\right]\right]$$

$$\sim\int\limits_{x(\beta)=x(0)}\mathcal{D}x\exp\left[-\int_0^\beta d\tau\left(\frac{m}{2}\dot x(\tau)^2+V(x(\tau))\right)\right]$$

and ditto with insertions in G_β convention $\lim\limits_{N\to\infty}\int\prod\limits_{j=1}^N\frac{dx}{2\pi}\to\int\mathcal{D}x$

Wick-rotation back to Minkowski:

$$\tau = it, \quad d\tau = i\,dt, \quad \left(\mathring{x}(\tau)\right)^2 = \left(\frac{dx}{d\tau}\right)^2 = -\frac{dx}{dt}$$

$$\Rightarrow \quad -\frac{m}{2}\mathring{x}(\tau)^2 - V(x) = \frac{m}{2}\left(\frac{dx}{dt}\right)^2 - V(x) = \mathcal{L}_M(x,\mathring{x})$$ Lagrange funct. in Minkowski

$$\Rightarrow \quad G_T^{(n)}(t_1,\dots,t_n) = \frac{\int \mathcal{D}x\; x(t_1)\cdots x(t_n)\, \exp\left\{ i\int_0^\beta dt\, \mathcal{L}_M(x,\mathring{x})\right\}}{\int \mathcal{D}x\, \exp\{ \text{— " —} \}}$$

(for $H = \frac{p^2}{2m} + V(x)$) contr. example [R] ch. 5 p 163

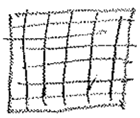
- Similarly one can consider the transition amplitude

Schrödinger pic $\quad \langle t', s | t; s\rangle \to \int \mathcal{D}x\, \exp\left\{ i\int_t^{t'} ds\, \mathcal{L}_M(x(s),\dot{x}(s))\right\}$ [R] ch. 5, [IZ] ch. 9.

Generalisation QM in 1D to QFT in 3D

recall $\quad t \to x^\mu$ $\quad$ in class. field theory
$\quad x(t) \to \phi(x^\mu)$

- the discretisation we did for a scalar $x(t)$ can be done for a vector $x \to \vec{x}$
- we can consider a field on discretised space as a vector

e.g. 2D  $\phi(t,\vec{x}_n) = \vec{X}(t)$, $\quad [\vec{X}(t)]_n = \phi(t,\vec{x}_n)$
nth component

$\to$ replace $x(t) \to \phi(x^\mu)$, $\int dt\, \mathcal{L}_M \to \int d^4x\, \mathcal{L}_M$

$$\Rightarrow \quad G_T^{(n)}(x_1,\dots,x_n) = \frac{\int \mathcal{D}\phi\; \phi(x_1)\cdots\phi(x_n)\, \exp\left\{ i\int d^4x\, \mathcal{L}_M(\phi,\partial_\mu\phi)\right\}}{\int \mathcal{D}\phi\, \exp\{ \text{— " —} \}}$$

with $\quad \mathcal{L}_M = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi)$