2.2 Wick's Theorem and the Schwinger propagator in the path-integral formalism

- as a first step we need to verify that we obtain the same Green functions and rules for perturbation theory in the path integral formalism as before
- we shall also prove Wick's theorem in this setting

In the following we will work in Euclidean space, after Wick rotation

$$\mathcal{L}_M = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - V(\phi) = \frac{1}{2}(\partial_t\phi)^2 - \sum_{i=1,2,3}\frac{1}{2}(\partial_i\phi)^2 - V(\phi)$$

$$\mathcal{L}_E = -\mathcal{L}_M(t=-i\tau)$$

$$= \frac{1}{2}(\partial_\tau\phi)^2 + \frac{1}{2}\sum_{i=1,2,3}(\partial_i\phi)^2 + V(\phi) \equiv \frac{1}{2}(\partial_\mu\phi)^2 + V(\phi)$$

Einstein sum. conv. with new convention $(\partial_\mu\phi)^2 = \partial_\mu\phi\,\partial_\mu\phi$ with Euclidean metric

$$S_E \equiv \int d\tau \int d^3\vec{x}\,\mathcal{L}_E$$

time interval to be specified (below $[0,\beta]$)

last lecture $\Rightarrow$ $G_\beta^{(n)}(x_1,..,x_n) \equiv \dfrac{\int\mathcal{D}\phi\,\phi(x_1)..\phi(x_n)\,e^{-S_E}}{\int\mathcal{D}\phi\,e^{-S_E}}$

$$\equiv \langle\phi(x_1)..\phi(x_n)\rangle$$

- no operators, no time order, convergent integrals!

Perturbation theory: $S_E = S_0 + S_{int}$, S_0 quadratic (Klein-Gordon)

define $\langle\mathcal{O}\rangle_0 \equiv \dfrac{\int\mathcal{D}\phi\,\mathcal{O}\,e^{-S_0}}{\int\mathcal{D}\phi\,e^{-S_0}}$

$$\Rightarrow G_\beta^{(n)}(x_1,..,x_n) = \frac{\langle\phi(x_1)..\phi(x_n)\,e^{-S_{int}}\rangle_0}{\langle e^{-S_{int}}\rangle_0}$$

do Taylor expansion, same as for $G_T^{(n)} = \dfrac{\langle 0|T\{\phi(x_1)..\phi(x_n)\hat{S}\}|0\rangle}{\langle 0|\hat{S}|0\rangle}$

Proof of Wick's Theorem

- We will formulate Wick in the new path integral formalism, showing that objects $\langle \phi(x_1) \dots \phi(x_n) \rangle_0$ can be reduced to contractions $\langle \phi(x_1) \phi(x_2) \rangle_0$ of only 2 fields.
- In order to do so
  - * we discretise again and collect $\phi(x)$ for all points x into a vector $\vec{v}$
  - * we then write $S_0 = \frac{1}{2} \vec{v}^T A \vec{v}$ where A is a matrix. Furthermore we assume that A is symmetric and invertible : $A = A^T$, $\exists\, A^{-1}$ s.th. $AA^{-1} = \mathbb{1}$
  - * we introduce a source $\vec{J}$: $S_0 \to S_0 - \vec{J}^T \vec{v}$ to generate expectation values [this trick goes back to J. Schwinger]

[the same can be formulated in continuum language (later) where the scalar product is
$\frac{1}{2} \vec{v}^T A \vec{v} - \vec{J}^T \vec{v} = \frac{1}{2} \int dx\, dy\, \phi(x) A(x,y) \phi(y) - \int dx\, J(x) \phi(x)$]

Define $$e^{W(\vec{J})} \equiv \int d\vec{v}\, e^{-\frac{1}{2} \vec{v}^T A \vec{v} + \vec{J}^T \vec{v}}$$

; $\underline{\text{shift}}$ $\vec{v} \to \vec{v} + A^{-1} \vec{J}$

$\Rightarrow$ $\vec{v}^T \to \vec{v}^T + \vec{J}^T A^{-1^T} = \vec{v}^T + \vec{J}^T A^{-1}$

$$= \int d\vec{v} \exp\left[-\frac{1}{2} (\vec{v}^T + \vec{J}^T A^{-1}) A (\vec{v} + A^{-1} \vec{J}) + \vec{J}^T (\vec{v} + A^{-1} \vec{J}) \right]$$

$$= \int d\vec{v} \exp\left[-\frac{1}{2} \vec{v}^T A \vec{v} - \frac{1}{2} \vec{v}^T \vec{J} - \frac{1}{2} \vec{J}^T \vec{v} - \frac{1}{2} \vec{J}^T A^{-1} \vec{J} + \vec{J}^T \vec{v} + \vec{J}^T A^{-1} \vec{J} \right]$$

$$= e^{+\frac{1}{2} \vec{J}^T A^{-1} \vec{J}} \int d\vec{v}\, e^{-\frac{1}{2} \vec{v}^T A \vec{v}}$$ using $\vec{J}^T \vec{v} = \vec{v}^T \vec{J}$

in components
(ex 5.3. compute $e^{W(0)}$) $$e^{W(\vec{J})} = \int d\vec{v}\, e^{-\frac{1}{2} v_i A_{ij} v_j + J_i v_i} = e^{\frac{1}{2} J_i A^{-1}_{ij} J_j} \int d\vec{v}\, e^{-\frac{1}{2} v_i A_{ij} v_j}$$

$$\Rightarrow \quad \langle V_i V_j \cdots V_k \rangle_0 = \frac{\int d\vec{V}\, V_i V_j \cdots V_k\, e^{-\frac{1}{2} V_n A_{nm} V_m}}{\int d\vec{V}\, e^{-\frac{1}{2} V_n A_{nm} V_m}}$$

$$\underline{\underline{\text{l.h.s}}} \quad \frac{\frac{d}{dJ_i}\frac{d}{dJ_j}\cdots\frac{d}{dJ_k} e^{W(\vec{J})}}{e^{W(0)}}\Bigg|_{\vec{J}=0} \overset{\text{rhs}}{=} \frac{d}{dJ_i}\frac{d}{dJ_j}\cdots\frac{d}{dJ_k} e^{\frac{1}{2} J_n A^{-1}_{nm} J_m}\Bigg|_{\vec{J}=0}$$

$$= \frac{d}{dJ_i}\frac{d}{dJ_j}\cdots\frac{d}{dJ_k}\left[1 + \frac{1}{2} J_n A^{-1}_{nm} J_m + \frac{1}{2!}\left(\frac{1}{2} J_{n_1} A^{-1}_{n_1 m_1} J_{m_1}\right)\left(\frac{1}{2} J_{n_2} A^{-1}_{n_2 m_2} J_{m_2}\right) + \dots\right]_{\vec{J}=0}$$

It holds:

- $\langle 1 \rangle_0 = 1$
- for an odd number of insertions we get zero $\langle V_1 \dots V_{2j+1} \rangle_0 = 0$
- $\langle V_i V_j \rangle_0 = (A^{-1})_{ij} \equiv \underbrace{V_i V_j}_{}$ (we will show what that is: $G_E^{(2)}$ ‖
- $\langle V_i V_j V_k V_e \rangle_0 \overset{\uparrow}{=} \underbrace{V_i V_j}_{}\ \underbrace{V_k V_e}_{} + \underbrace{V_i \overline{V_j}}_{}\overline{V_k V_e} + \underbrace{V_i \overline{V_j V_k} V_e}_{}$

  ex. 5.2

- Wick's Theorem

$$\langle V_1 \dots V_{2n} \rangle_0 = \sum_{\substack{\text{all}\\ \text{combinations}}} \underbrace{V_{i_1} V_{i_2}}_{} \cdots \underbrace{V_{i_{2n-1}} V_{i_{2n}}}_{}$$

why multiplicity 1:

$$\prod_{e=1}^{2n} \frac{d}{dJ_e} \sum_{j=0}^{\infty} \frac{1}{j!}\left(\frac{1}{2} J_p A^{-1}_{pq} J_q\right)^j \Bigg|_{J=0} = \prod_{e=1}^{2n} \frac{d}{dJ_e} \frac{1}{n!}\left(\frac{1}{2} J_p A^{-1}_{pq} J_q\right)^n$$

- factor $\frac{1}{2}$: $\frac{d}{dJ_e} \frac{1}{2} J_p A^{-1}_{pq} J_q = \frac{1}{2}\left(A^{-1}_{eq} J_q + J_p A^{-1}_{pe}\right) \overset{A = A^T}{=} A^{-1}_{eq} J_q$ removes $\frac{1}{2}$

$$\frac{d}{dJ_k} A^{-1}_{eq} J_q = A^{-1}_{ek} \quad \text{no num. factor}$$

- factorial: $\frac{d}{dJ_e}\left(\frac{1}{2} J_p A^{-1}_{pq} J_q\right)^n = n\, A^{-1}_{pq} J_q \left(\frac{1}{2} J_{p'} A^{-1}_{p'q'} J_{q'}\right)^{n-1}$

$\Rightarrow$ • $\frac{d}{dJ}$ acting on powers of the bilinear $\frac{1}{n!}\left(\frac{1}{2} J A^{-1} J\right)^n$ removes $\frac{1}{2}$ and reduces factoral $\frac{1}{n!} \to \frac{1}{(n-1)!}$

• $\frac{d}{dJ}$ actn on linear $J A^{-1}$ adds no factors

• all J has to differentiated away ($\vec{J}=0$ at end) $\Rightarrow$ no $\frac{1}{2}$, no factorial left

[alternative proof : induction]

<u>the propagator $\langle \phi(x)\, \phi(y) \rangle_0$ in the continuum :</u>

claim $\langle V_i V_j \rangle_0 = A^{-1}_{ij}$ cont. is the Euclidean Green's function

We have $S_0 = \int dx^4 \left\{ \frac{1}{2} \partial_\mu \phi \partial_\mu \phi + \frac{m^2}{2} \phi(x) \right\}$ $\qquad d^4x = d\tau d\vec{x}^3$ Euclidean

$= \int dx^4 \left\{ \frac{1}{2} \phi(x) \left\{ -\partial_\mu \partial_\mu + m^2 \right\} \phi(x) \right\}$

$= \frac{1}{2} \int dx^4 \int dy^4\, \phi(x)\, A(x,y)\, \phi(y)$ $\quad \left(\hat{=} \frac{1}{2} \vec{v} A \vec{v} \right.$ 1 product = 1 $\left.\int\right)$

with $A(x,y) = \delta^4(x-y) \left(-\frac{\partial^2}{(\partial x^\mu)^2} + m^2 \right)$

We need $= \int dy^4\, A(x,y)\, A^{-1}(y,z) = \delta^{(4)}(x-z)$ $\qquad \left(\hat{=} A_{ij} (A^{-1})_{jk} = \delta_{ik} \right)$

Solution
$$\boxed{A^{-1}(y,z) = \int \frac{dp^4}{(2\pi)^4} \frac{1}{p^2+m^2} e^{i p_\mu (y_\mu - z_\mu)} = G_E(y,z)}$$
$\uparrow$ page (18)

as $= \int dy^4\, \delta^4(x-y) \left(-\frac{\partial^2}{(\partial x^\mu)^2} + m^2 \right) A^{-1}(y,z)$

$= \left(-\frac{\partial^2}{(\partial y^\mu)^2} + m^2 \right) A^{-1}(y,z) = \int \frac{dp^4}{(2\pi)^4} \frac{p^2+m^2}{p^2+m^2} e^{i p_\mu (y_\mu - z_\mu)} = \delta^{(4)}(y-z)$

<u>Note</u>: in this approach we could have derived Wick in the continuum:

$$e^{W(J)} = \int \mathcal{D}\phi \, \exp\left[-\frac{1}{2}\int d^4x\, d^4y\, \phi(x) A(x,y) \phi(y) + \int d^4x\, J(x) \phi(x)\right]$$

shift $\phi(x) \to \phi(x) + \int d^4y\, A^{-1}(x,y) J(y)$

using $\int d^4y\, A(x,y) A^{-1}(y,z) = \delta^{(4)}(x-z)$

$$\Rightarrow \text{rhs} = e^{\frac{1}{2}\int d^4x\, d^4y\, J(x) A^{-1}(x,y) J(y)} \int \mathcal{D}\phi\, e^{-\frac{1}{2}\int\int \phi A \phi}$$

generate ϕ's by functional derivative:

$$\frac{\delta}{\delta J(y)} J(x) = \delta^{(4)}(x-y)$$

$$\Rightarrow \langle \phi(x_1) \ldots \phi(x_n) \rangle_0 = \frac{\frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)}\, e^{W(J)}}{e^{W(J=0)}}$$

Appendix

Check: in Fourier space at finite volume:

4-Volume $V = L_0 L_1 L_2 L_3$, periodic boundary cond (in all directions)

$$\phi(x_\mu + L_\mu) = \phi(x_\mu) \qquad \tilde{\phi}(p_\mu) \in \mathbb{C}, \ \phi(x) \in \mathbb{R}$$

$$\Rightarrow \boxed{\phi(x) = \frac{1}{V} \sum_{p_\mu} \tilde{\phi}(p_\mu)\, e^{i p_\mu x_\mu}}, \quad p_\mu = \frac{2\pi}{L_\mu} n_\mu, \ n_\mu \in \mathbb{Z}, \ \mu = 0,1,2,3$$

inverse trafo: $\tilde{\phi}(p) = \frac{1}{V} \int dx^4\, e^{-i p_\mu x_\mu} \phi(x)$

it holds (ex 5.4) that in the continuum limit $V \cdot \delta_{p_\mu + Q_\mu, 0} \to (2\pi)^4 \delta^{(4)}(p+Q)$

$$\frac{1}{V} \sum_{p_\mu \in \mathbb{Z}^4} \tilde{\phi}(p_\mu)\, e^{i p_\mu x_\mu} \to \int \frac{dp^4}{(2\pi)^4} \tilde{\phi}(p)\, e^{i p \cdot x}$$

we have

$$\langle \phi(x) \phi(y) \rangle_0 = \frac{1}{V} \sum_{p_\mu} \frac{1}{V} \sum_{Q_\mu} \langle \tilde{\phi}(p_\mu) \tilde{\phi}(Q_\mu) \rangle_0$$

if we can show that $\boxed{\langle \tilde{\phi}(p_\mu) \tilde{\phi}(Q_\mu) \rangle_0 \overset{!}{=} \frac{V \delta_{p_\mu + Q_\mu, 0}}{p_\mu^2 + m^2}}$ $\Rightarrow$ previous page

- $\phi(x) = \phi^*(x) \Rightarrow \tilde{\phi}^*(p_\mu) = \tilde{\phi}(-p_\mu)$ real scalar field

so writing $\tilde{\phi}(p_\mu) = a(p_\mu) + i b(p_\mu) \Rightarrow a(-p_\mu) = a(p_\mu)$, $b(-p_\mu) = -b(p_\mu)$

((only $\frac{1}{2}$ of the Fourier comp. are indep, choose $p_0 > 0$ ones))

- $$S_0 = \frac{1}{2} \int dx^4 \left(\partial_\mu \phi \partial_\mu \phi + m^2 \phi^2\right) \overset{F.T.}{=} \frac{1}{2} \underbrace{\int dx^4}_{} \frac{1}{V^2} \sum_{p,Q} \left\{ i p_\mu \tilde{\phi}(p)\, i Q_\mu \tilde{\phi}(Q) + m^2 \tilde{\phi}(p) \tilde{\phi}(Q) \right\} \underbrace{e^{i(p+Q)x}}_{V \delta_{Q+p,0}}$$

$$= \frac{1}{2} \frac{1}{V} \sum_p \underbrace{\tilde{\phi}(-p)}_{= \tilde{\phi}^*(p)} (p_\mu^2 + m^2) \tilde{\phi}(p) = \frac{1}{V} \sum_{p,\, p_0 > 0} (p_\mu^2 + m^2) \left[a^2(p) + b^2(p) \right]$$

Now: $\langle \tilde{\phi}(p_\mu) \tilde{\phi}(Q_\mu) \rangle_0 = \langle a(p) a(Q) + i b(p) a(Q) + i a(p) b(Q) - b(p) b(Q) \rangle_0$

we only have Gaussian integrals $\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$, $\int_{-\infty}^{\infty} dx\, x^2 e^{-ax^2} = \frac{1}{2a} \sqrt{\frac{\pi}{a}}$

$$\langle \tilde{O} \rangle_0 \overset{p\text{-space}}{=} \frac{\prod_{\substack{p_\mu \\ p_0>0}} \int da(p_\mu)\, db(p)\, \tilde{O}(a,b)\, e^{-\frac{1}{V}\sum_{\substack{p_\mu \\ p_0>0}} (p_\mu^2+m^2)\left[a^2(p)+b^2(p)\right]}}{\prod_{\substack{p_\mu \\ p_0>0}} \int da(p_\mu)\, db(p_\mu)\, e^{\text{---}\;''\;\text{---}}}$$

(recall the $a(p_\mu)$, $b(p_\mu)$ are functions in p-space, not operators)

$$\Rightarrow \quad \langle a(p_\mu)\, b(Q_\mu) \rangle_0 = \langle b(p_\mu)\, a(Q_\mu) \rangle_0 = 0$$

$$\langle a(p_\mu)\, a(Q_\mu) \rangle_0 = \frac{V}{2(p_\mu^2+m^2)} \left[\delta_{p,Q} + \delta_{p,-Q} \right]$$

$$\langle b(p)\, b(Q) \rangle_0 = \frac{V}{2(p_\mu^2+m^2)} \left[\delta_{p,Q} - \delta_{p,-Q} \right]$$

$$\Rightarrow \quad \langle \tilde{\phi}(p)\, \tilde{\phi}(Q) \rangle_0 = \frac{V\, \delta_{p,-Q}}{p_\mu^2+m^2} = \frac{V\, \delta_{p+Q,0}}{p_\mu^2+m^2}$$

as we wanted to show.

Literature [PS] ch 9.2