2.3 Schwinger - Dyson Equations and generating functionals

- using the path integral formalism we shall derive some relations that hold non-perturbatively. In preparation we need

- functional derivatives:

a functional $F[\phi]$ depends on a function ϕ just as the function $\phi(x)$ depends on a variable x

example: $S_E[\phi] = \int dx^4 \mathcal{L}_E(\phi(x), \partial_\mu \phi(x))$

Def functional derivative $\boxed{\dfrac{\delta F[\phi]}{\delta \phi(y)} \equiv \lim_{\varepsilon \to 0} \dfrac{F[\phi(\cdot) + \varepsilon \delta(y - \cdot)] - F[\phi]}{\varepsilon}}$

example: $F[\phi] = \int dx^4 \phi(x) \Rightarrow \dfrac{\delta F[\phi]}{\delta \phi(y)} = \lim_{\varepsilon \to 0} \dfrac{\int d^4x \left((\phi(x) + \varepsilon \delta^{(4)}(y-x)) - \phi(x) \right)}{\varepsilon} = 1$

ditto $F[\phi] = \int dx^4 \phi^n(x) \Rightarrow \dfrac{\delta F[\phi]}{\delta \phi(y)} = n \phi^{n-1}(y)$

$$\Rightarrow \frac{\delta S_E[\phi]}{\delta \phi(y)} \overset{\text{Taylor}}{=} \int d^4_x \frac{\partial \mathcal{L}_E(\phi(x), \partial_\mu \phi(x))}{\partial \phi(x)} \delta^{(4)}(x-y) = \frac{\partial \mathcal{L}_E}{\partial \phi(y)}$$

note: when differentiating we do not treat $\phi, \partial_\mu \phi$ as indep variables but differentiate both wrt ϕ:

e.g. for $\mathcal{L}_E = \frac{1}{2} \partial_\mu \phi(x) \partial_\mu \phi(x) + \frac{m^2}{2} \phi(x)^2 + \frac{g}{3!} \phi^3(x) + \frac{\lambda}{4!} \phi^4(x)$

we get $\dfrac{\delta S_E[\phi]}{\delta \phi(y)} = -\frac{1}{2} \partial_\mu \partial_\mu \phi(y) + m^2 \phi(y) + \frac{g}{2!} \phi^2(y) + \frac{\lambda}{3!} \phi^3(y)$

Schwinger - Dyson Equations:

These follow from "trivial" identities, such as integrals of total derivatives

- discrete: $0 = \int_{-\infty}^{\infty} d\vec{v} \, \frac{d}{dv_k}\left\{ e^{-S_E(\vec{v})} \right\}_1$, $0 = \int d\vec{v} \frac{d}{dv_k}\left\{ v_i v_j \ldots e^{-S_E(\vec{v})} \right\}$ for b.c. $S(\vec{v}) \to \infty$, $|\vec{v}| \to \infty$

- continuous: $\boxed{0 = \int \mathcal{D}\phi \frac{\delta}{\delta\phi(x)} \left\{ e^{-S_E[\phi]} \right\}_1}$ $\hat{=} I$, $0 = \int \mathcal{D}\phi \frac{\delta}{\delta\phi(x)} \left\{ \phi(y)\phi(z) \ldots e^{-S_E[\phi]} \right\}$

for appropriate b.c.

back to our example:

$$I: \quad 0 = \int \mathcal{D}\phi \left\{ \left[-\partial_\mu^2 + m^2 \right] \phi(x) + \frac{\lambda}{3!} \phi^3(x) \right\} e^{-S_E[\phi]}$$

$\cdot \left[\int \mathcal{D}\phi \, e^{-S_E} \right]^{-1}$

$$\Rightarrow \quad 0 = \left[-\partial_\mu^2 + m^2 \right] \langle \phi(x) \rangle + \frac{\lambda}{3!} \langle \phi^3(x) \rangle + \frac{g}{2} \langle \phi^2(x) \rangle$$

classical equations of motion hold for expectation values!

- get all identities: use generating functional $Z[J]$

Def $Z[J] = \int \mathcal{D}\phi \, e^{-S_E[\phi] + \int d^4x \, J(x) \phi(x)}$ including source $J(x)$

this can be used to express all the above identities:

- we have $\frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} = \int \mathcal{D}\phi \, \phi(x_1) \ldots \phi(x_n) \, e^{-S_E[\phi] + \int d^4x \, J(x) \phi(x)}$

i.e. $G_E^{(n)}(x_1, \ldots, x_n) = \langle \phi(x_1) \ldots \phi(x_n) \rangle = \frac{\left. \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right|_{J=0}}{Z[0]}$

- Normalise $Z[0] \equiv 1$

$$\Rightarrow \quad Z[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \ldots dx_n \left. \frac{\delta^n Z[J]}{\delta J(x_1) \ldots \delta J(x_n)} \right|_{J=0} J(x_1) \ldots J(x_n)$$ Taylor expansion

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \ldots dx_n \, G_E^{(n)}(x_1, \ldots, x_n) \, J(x_1) \ldots J(x_n)$$

is the generating functional for all Green functions

Schwinger – Dyson eq for $Z[J]$

$$0 = \int \mathcal{D}\phi \, \frac{\delta}{\delta\phi(x)} \left\{ e^{-S_E[\phi] + \int d^4y\, J(y)\phi(y)} \right\}$$

$$= \int \mathcal{D}\phi \left\{ -\frac{\partial \mathcal{L}_E}{\partial \phi(x)} + J(x) \right\} e^{-S_E[\phi] + \int d^4y\, J(y)\phi(y)} \qquad (*)$$

- we can generate any polynomial $(\phi(x))^n$ by $\frac{\delta^n}{(\delta J)^n}$:

$$\int \mathcal{D}\phi \, (\phi(x))^n e^{-S_E + \int J\phi} = \int \mathcal{D}\phi \left[\frac{\delta}{\partial J(x)}\right]^n e^{-S_E + \int J\phi}$$

and if we consider $\frac{\partial \mathcal{L}_E}{\partial \phi(x)} \equiv \mathcal{L}_E'(\phi(x))$ as a polynomial in $\phi(x)$ (→ exp)

we have $(*)$: $0 = \int \mathcal{D}\phi \underbrace{\left\{ -\mathcal{L}_E'\left(\frac{\delta}{\delta J(x)}\right) + J(x) \right\}}_{\phi(x)\text{-indep}} e^{-S_E + \int J\phi} \Leftrightarrow \boxed{0 = \left[-\mathcal{L}_E'\left(\frac{\delta}{\delta J(x)}\right) + J(x)\right] Z[J]}$

S-D. eq.

- our example:

$$\mathcal{L}_E(\phi(x)) = \frac{1}{2}\phi\Delta^{-1}\phi(x) + \frac{g}{3!}\phi^3(x) + \frac{\lambda}{4!}\phi^4(x), \quad \Delta^{-1} \equiv -\partial_\mu^2 + m^2$$

$$\Rightarrow \mathcal{L}_E'(\phi) = \Delta^{-1}\phi(x) + \frac{g}{2!}\phi^2(x) + \frac{\lambda}{3!}\phi^3(x)$$

$$\Rightarrow \text{S.D. eq}: -\left[\Delta^{-1}\frac{\delta}{\delta J(x)} + \frac{g}{2}\left(\frac{\delta}{\delta J(x)}\right)^2 + \frac{\lambda}{3!}\left(\frac{\delta}{\delta J(x)}\right)^3 - J(x)\right] Z[J] = 0$$

- Graphical representation

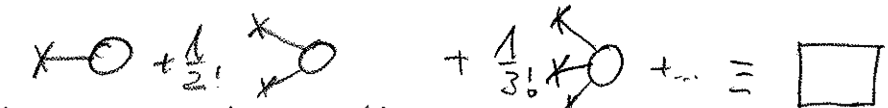
Use that $Z[J]$ generates all $G_E^{(n)}$ (decorated with $J(x_1) \dots J(x_n)$)

denote $J(x) \equiv \times$, $G^{(n)} \equiv$

connected parts

conn.- and discon. parts

$\Rightarrow Z[J] = 1 +$

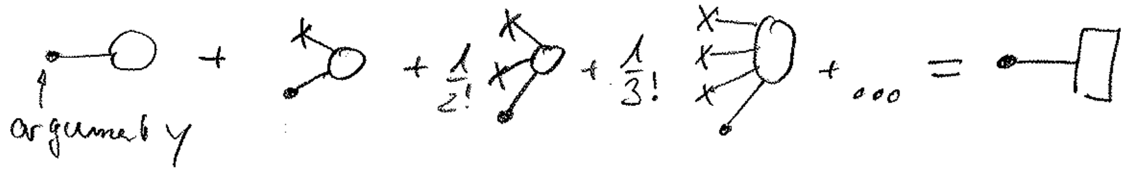


$\int J(x) G^{(1)}(x)$ $\iint J(x)J(y) G^{(2)}(x,y)$

$\Delta(x,y) = \langle \phi(x)\phi(y)\rangle$ is the Euclidean Green's function

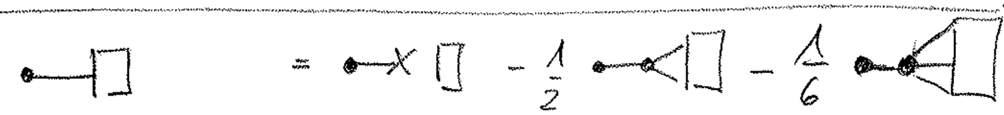
$\Rightarrow \quad \Delta^{-1}\Delta = \mathbb{1}$

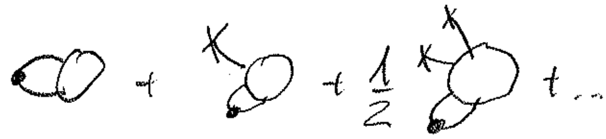
denote $\Delta(x,y) =$  , $g \equiv$  , $\lambda \equiv$ 

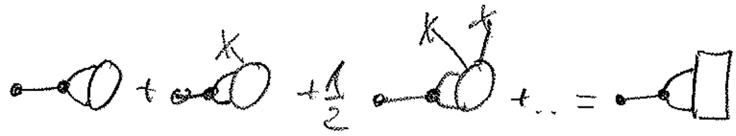
• $\dfrac{\delta Z[J]}{\delta J(y)} =$ 

S-D eq: $\displaystyle \Delta^{-1}\frac{\delta}{\delta J(x)} Z[J] = \left[-\frac{g}{2}\left(\frac{\delta}{\delta J}\right)^2 - \frac{\lambda}{6}\left(\frac{\delta}{\delta J}\right)^3 + J\right] Z[J]$

$|\cdot\Delta \Rightarrow \boxed{\displaystyle \frac{\delta}{\delta J(x)} Z[J] = \Delta\left[J - \frac{g}{2}\left(\frac{\delta}{\delta J}\right)^2 - \frac{\lambda}{6}\left(\frac{\delta}{\delta J}\right)^3\right] Z[J]}$

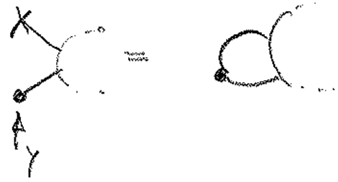


as $\displaystyle \frac{\delta^2}{(\delta J(x))^2} Z[J] =$ 

$\displaystyle \Rightarrow \Delta \frac{\delta^2}{(\delta J)^2} Z =$ 

etc.

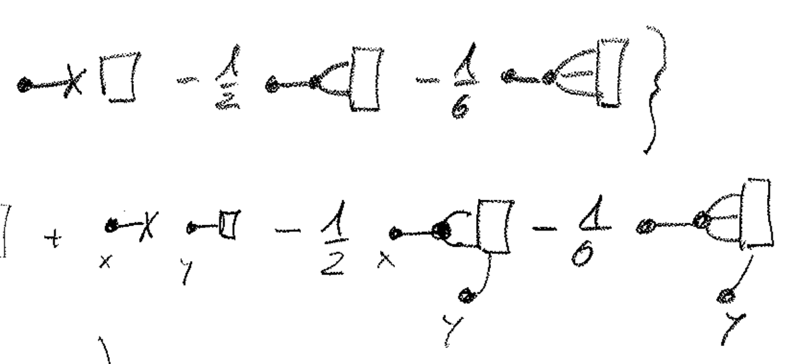
($\frac{\delta}{\delta J(y)}$ deletes a × and joins it with the other endpoint

with argument y $\quad \frac{\delta}{\delta J}$ )

we can now iterate by reinserting the S.-D. eq into itself:

evaluate $\Delta \cdot \frac{g}{2}\left(\frac{\delta}{\delta J}\right)^2$ - term by reiterating the S.D.:

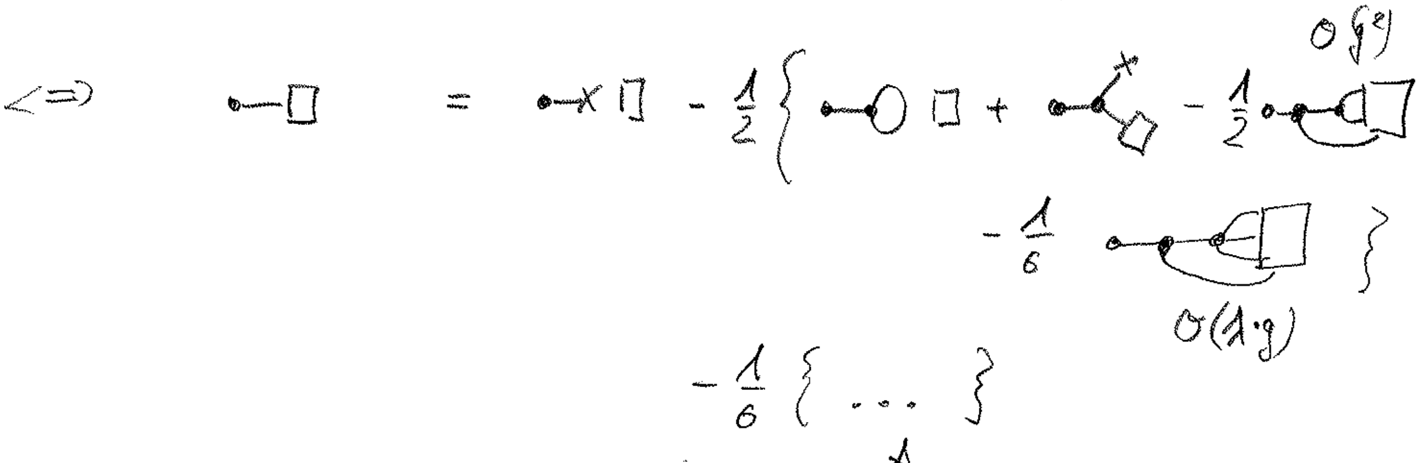
$$\frac{\delta}{\delta J(y)}\frac{\delta}{\delta J(x)} Z[J] = \frac{\delta}{\delta J(y)}\Big\{ \ldots \Big\}$$



- take the limit $x \to y$ (merge) and multiply with Δ

use above

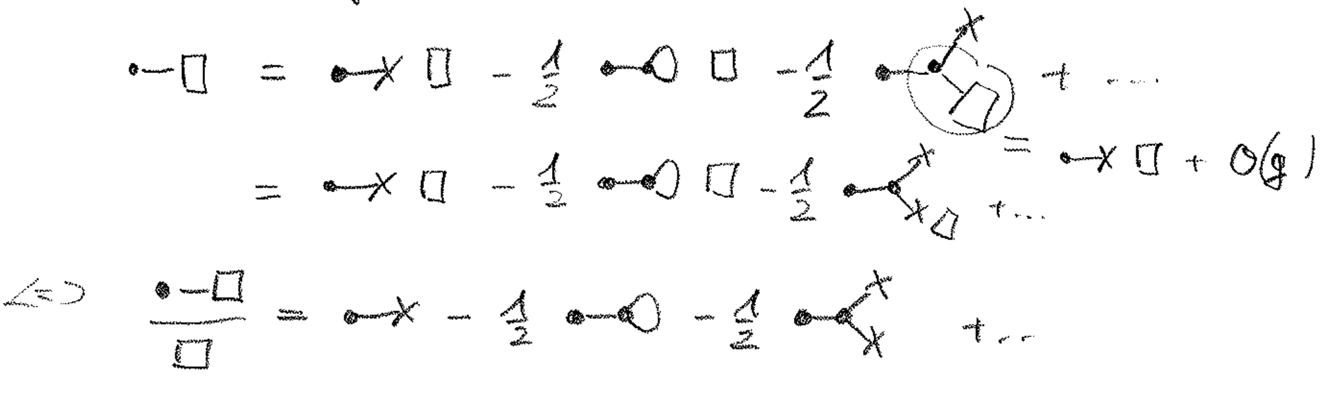
$$\Rightarrow \quad \frac{\delta}{\delta J(x)} Z[J] = \Delta\left[J - \frac{g}{2}\left(\frac{\delta}{\delta J(x)}\right)^2 - \frac{\lambda}{6}\left(\frac{\delta}{\delta J(x)}\right)^3\right] Z[J]$$



* here we may use another iteration for $\frac{\delta^3}{\delta J^3}$

- if we want to return to a perturbative expansion we can drop higher order terms after iteration:

to order $\mathcal{O}(g^2)$, $\mathcal{O}(\lambda)$



Other generating functionals

A) $\boxed{W[J] = \ln\left(Z[J]\right)} \Leftrightarrow Z[J] = e^{W[J]}$

- generates all connected Green's functions $G^{(n)}_{E,C}$

$$W[J] = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1 \ldots dx_n \, G^{(n)}_{E,C}(x_1 \ldots x_n) \, J(x_1) \ldots J(x_n)$$

(proof to order 4 : ex 6.1)

- S.D. eq for $W[J]$: $\quad f_E'\left(\frac{\delta W[J]}{\delta J(x)} + \frac{\delta}{\delta J(x)} \right) \mathbb{1} = J(x)$

(with $\frac{\delta}{\delta J(x)} \mathbb{1} = 0$)

B) Legendre transform $\quad \varphi(x) \equiv \frac{\delta W[J]}{\delta J(x)} \Rightarrow$

$$\boxed{\Gamma[\varphi] \equiv W[J] - \int d^4x \, \varphi(x) J(x)}$$

$\Rightarrow \quad \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)} = -J(x) \quad$; if we set $J \equiv 0$ at the end $\frac{\delta \Gamma}{\delta \varphi} = 0$ is the eq. for an extremum

- S.D. eqs for $\Gamma \to$ ex 6.2
- Taylor $\Gamma[\varphi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int dx_1^4 \ldots dx_n^4 \, \Gamma^{(n)}_E(x_1 \ldots x_n) \, \varphi(x_1) \ldots \varphi(x_n)$

generates the 1-particle-irreducible (1PI) Green's funct.:

$G^{(n)}_{E,C} =$  , $A^{(n)}_{E,C} =$  may contain 

1PI $=$  does not contain diagrams that become disconnected when cutting 1 line