3. Renormalisation

3.1. Power counting and regularisation

We consider the ϕ^4 as an example:

$J_E = \frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$ is positive for $m^2, \lambda > 0$ (and sym $\phi \to -\phi$)

- $J_E \xrightarrow[|\phi|\to\infty]{} \infty$ seems to suggest that apparently

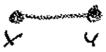
$$\langle \phi(x_1)\ldots\phi(x_n)\rangle = \frac{\int \mathcal{D}\phi\, \phi(x_1)\ldots\phi(x_n)\, e^{-\int d^4x\, J_E}}{\int \mathcal{D}\phi\, e^{-\int d^4x\, J_E}} \quad \text{converges !?}$$

- this is not always the case, as the 2-point function shows: expand $\langle\phi(x)\phi(y)\rangle$ $(= \langle\phi(x)\phi(y)\rangle_c)$ to $O(\lambda)$

$$= \langle\phi(x)\phi(y)\rangle_0 - \frac{\lambda}{4!}\int d^4z \left[\langle\phi(x)\phi(y)\phi^4(z)\rangle_0 - \langle\phi(x)\phi(y)\rangle_0\langle\phi^4(z)\rangle_0\right]$$

with $\langle \phi(x)\,\overbrace{\phi(y)\,\phi(z)}^{3}\,\phi(z)\,\phi(z)\phi(z)\rangle_0$, $\underbrace{\phi(x)\,\phi(y)\,\phi(z)}_{4}$, $\underbrace{\phi(z)\phi(z)}_{1}$

$$= \langle\phi(x)\phi(y)\rangle_0 - \frac{\lambda}{2}\int d^4z\, \langle\phi(x)\phi(z)\rangle_0\langle\phi(y)\phi(z)\rangle_0\langle\phi(z)\phi(z)\rangle_0$$

Graphically  + 

- we have $\langle\phi(x)\phi(y)\rangle_0 = \int_p \frac{e^{ip\cdot(x-y)}}{p^2+m^2}$, $\int_p \equiv \int\frac{d^4p}{(2\pi)^4}$ here

and for $x \neq y$ this integral is finite (cf 6.3)

- $\langle\phi(z)\phi(z)\rangle_0$ is z-indep so consider next

$$= \int d^4z\, \langle\phi(x)\phi(z)\rangle_0\langle\phi(y)\phi(z)\rangle_0 = \int d^4z \int_p\int_Q \frac{e^{iP(x-z)}\, e^{iQ(y-z)}}{(p^2+m^2)(Q^2+m^2)}$$

$\int d^4z$ gives $\delta^{(4)}(P+Q)$

$$= \int_p \frac{e^{iP(x-y)}}{(p^2+m^2)^2} = -\frac{d}{dm^2}\langle\phi(x)\phi(y)\rangle_0 \quad \text{which is again convergent}$$

- <u>But</u>: $\langle\phi(z)\phi(z)\rangle_0 = \int_p \frac{1}{p^2+m^2} \underset{\uparrow}{=} \int d\Omega_d \int_0^\infty d\rho\, \rho^{d-1}\frac{1}{\rho^2+m^2} = \infty$ as $\sim \int^\infty d\rho\, \rho = \infty$

polar coord in d dim

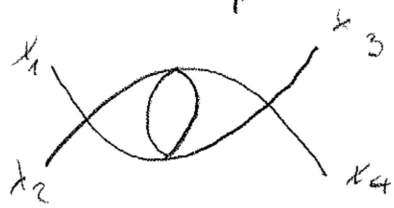
We see that loop corrections can be "∞", the reason being short distances $x=y$ for large momenta $|p| \gg m$. This is called ultraviolet (UV) physics.

- It is possible within perturbation theory to analyse which loop corrections may be divergent for arbitrary diagrams

Consider diagrams with:

N edges or vertices
E external legs
I internal lines
L loops
d space time (usually $d=4$)

example



has $N=4$, $E=4$
$I=6$, $L=3$

- superficial (*) degree of divergence D: inner line $\sim \frac{1}{p^2}$, loop $\int d^d p \Rightarrow$ $\boxed{D = d \cdot L - 2I}$
- for our example of a ϕ^4 theory we have

  * - every vertex N has 4 legs:
  - these are either external (E), or internal (I) which connect 2 vertices (incl. loop) $\Rightarrow 4N = E + 2I \Leftrightarrow \boxed{I = 2N - \frac{E}{2}}$

  * - every inner line I has 1 momentum
  - every vertex N has momentum-conservation (δ-function), removing 1 momentum-integral
  - global momentum-conserv. removed 1 further momentum-int.

  $\Rightarrow$ the # of loop momenta $\boxed{L = I - N + 1}$

$$\Rightarrow \boxed{D = d\left(2N - \frac{E}{2} - N + 1\right) - 2\left(2N - \frac{E}{2}\right) = d + (d-4)N + \left(1 - \frac{d}{2}\right)E}$$

<u>It follows</u>: • <u>for $d<4$</u>, e.g. $d=1,2,3$ $D \searrow$ with $N \nearrow$:

$d=3$: $D = 3 - N - \frac{E}{2}$, $d=2$: $D = 2-2N$

$\Rightarrow$ divergences may occur only for a small number of vertices or

• <u>for $d=4$:</u> $\boxed{D = 4 - E}$ so only divergences for $E \leq 4$?

(to ⊛) No: $E=6$ diagram may have $\Omega \to$ divergent subdiagrams. <u>Weinbergs theorem</u>: diagram is convergent if $D<0$ and D of all subgraphs is <0.

<u>Note:</u> these conclusions hold for our example of a ϕ^4-Theory

<u>Counting for ϕ^r-theory</u> $r \cdot N = E + 2I$,

$$\Rightarrow \boxed{D = d + \left(\frac{r}{2}(d-2) - d\right)N + \left(1 - \frac{d}{2}\right)E}$$

(in $d=2$ the counting is <u>indep</u> of r and E)

<u>General idea</u>: – we may be able to absorb the divergences

$E=2 \Rightarrow D=2$ in $\frac{m^2}{2}\phi^2$

$E=4 \Rightarrow D=0$ in $\frac{\lambda^4}{4!}\phi^4$

(because of the symmetry $\phi \to -\phi$ in our ϕ^4-theory $E = 1,3$ will not occur)

"renormalisation" of the coupling constants

Dimensional analysis and renormalisability

Q: Can we read off the perturbative renormalisability from the dimension of the coupling constant?

- counting in d dimensions: ℓ length

$S = \int d^d x \, \mathcal{L}$ is dimensionless $(\hbar = 1)$

- $[d^d x] = \ell^d \Rightarrow \boxed{[\mathcal{L}] = \ell^{-d}}$
- $[\partial_\mu] = \ell^{-1}$, $\partial_\mu \phi \partial_\mu \phi$-term in $\mathcal{L}$ $\Rightarrow \boxed{[\phi] = \ell^{1-\frac{d}{2}}}$

$\Rightarrow [G^{(n)}(x_1, \dots, x_n)] = \ell^{n(1-\frac{d}{2})}$

Consider an interaction $g_r \phi^r(x)$ in $\mathcal{L}$

setting $[g_r] = \ell^{-\delta}$ we have $-\delta + r(1-\frac{d}{2}) = -d$

$\Leftrightarrow \quad \boxed{\delta = d - \frac{r}{2}(d-2)}$

inserted in $\boxed{D = d - \delta N - (d-2)\frac{E}{2}}$

We only have a chance for pert. renorm. if D does <u>not</u> grow with the number of vertices N (or number of external legs E)

$\Rightarrow$ the coeff. have to satisfy $\delta \geq 0$, $(d-2) \geq 0$

examples

$g_4 \phi^4$: $\delta = 4 - d \geq 0 \Leftrightarrow 4 \geq d$

$g_3 \phi^3$: $\delta = 3 - \frac{d}{2} \geq 0 \Leftrightarrow 6 \geq d$

$g_6 \phi^6$: $\delta = 6 - 2d \geq 0 \Leftrightarrow 3 \geq d$ not renorm. in $d = 4$!

- We have to be systematic when dealing with divergences (subtracting ∞ or $\infty + 4$ both remove ∞ but change finite part)
- after regularising all has to be finite, at the end of the calculation we return to the question of removing the regulator

different regularisations

① Discretisation:

When we derived the path integral formalism we had to discretise anyhow, both space and time: $\Delta t = \varepsilon = \frac{t}{N}$, $\phi(t_1, \vec{x}_n) = \vec{X}_n$
$\Rightarrow \exists$ a minimal distance ε or $|\vec{x}_n - \vec{x}_m|$ $n \neq m$, so no UV diverg.

This approach is also called Lattice regularisation, it is math. rigorous but analytic calculations are difficult $\rightarrow$ num. approach

② cut-off regularisation

$$\int_{-\infty}^{\infty} d^4p \rightarrow \int_{|p| \leq \Lambda} d^4p \quad , \text{with } \Lambda \rightarrow \infty \text{ at the end}$$

However this breaks symmetries, in particular gauge-symmetries, is conceptually difficult [e.g. $|p| \leq \Lambda$, $|q| \leq \Lambda$ $\Rightarrow |p+q| \not\leq \Lambda$]

③ Pauli-Villars

introduction of a fictitious field with large mass M, $M \rightarrow \infty$ at the end, also problematic with gauge field

④ We will choose ...

... Dimensional Regularisation

- We will consider the dimension of space-time d as a parameter in $\mathbb{C}$, define rules of computation for that and analytically continue to $d=4$ at the end. Divergences: $\frac{1}{d-4}$

replace
$$\int_p = \int \frac{d^4p}{(2\pi)^4} \to \int \frac{d^dp}{(2\pi)^d}$$
$$\int dx^4 \to \int d^dx$$
$$(2\pi)^4 \delta^{(4)}(p) \to (2\pi)^d \delta^{(d)}(p)$$
$$\int d^4x\, e^{ipx} \to \int d^dx\, e^{ipx} = (2\pi)^d \delta^{(d)}(p)$$

rules of integration
i) $\int d^dp\, F(p+q) = \int d^dp\, F(p)$
ii) $\int d^dp\, F(\lambda p) = |\lambda|^{-d} \int d^dp\, F(p)$
iii) $\int d^dp \int d^{d'}q\, F(p) G(q) = \int d^dp\, F(p) \int d^{d'}q\, G(q)$

d-dim angular vol $c(d)$:

p d-dim vector $\int d^dp\, e^{-tp^2} \overset{(iii)}{=} \left(\int_{-\infty}^{\infty} dp_1\, e^{-tp_1^2} \right)^d = \left(\frac{\pi}{t}\right)^{\frac{d}{2}}$

polar coordinates $\int d^dp\, F(|p|) = c(d) \int_0^\infty d\rho\, \rho^{d-1} F(\rho)$

$$\Rightarrow \left(\frac{\pi}{t}\right)^{\frac{d}{2}} = \int d^dp\, e^{-tp^2} = c(d) \int_0^\infty d\rho\, \rho^{d-1} e^{-t\rho^2} = \frac{c(d)}{2t^{\frac{d}{2}}} \underbrace{\int_0^\infty \frac{dy}{\sqrt{y}}\, y^{\frac{d}{2}-\frac{1}{2}} e^{-y}}_{\Gamma\left(\frac{d}{2}\right)}$$

$t\rho^2 = y$, $2t\rho\, d\rho = dy$, $d\rho = \frac{dy}{2\sqrt{t}\sqrt{y}}$

$$\Leftrightarrow \boxed{c(d) = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}}$$

unusual consequence choose $F(p) = \frac{1}{|p|^\alpha}$ $\left(= e^{-\alpha \ln|p|} \quad \alpha \in \mathbb{C} \right)$

$$\Rightarrow \int d^dp\, F(\lambda p) = \int d^dp \frac{1}{|\lambda p|^\alpha} = \frac{1}{|\lambda|^\alpha} \int d^dp \frac{1}{|p|^\alpha} \overset{ii)}{=} \frac{1}{|\lambda|^d} \int d^dp \frac{1}{|p|^\alpha}$$

$\Rightarrow$ for $\alpha \neq d$ we have $\int d^dp \frac{1}{|p|^\alpha} = 0$!

((only certain integrals, e.g. over e^{ipx} exist – still the method works))