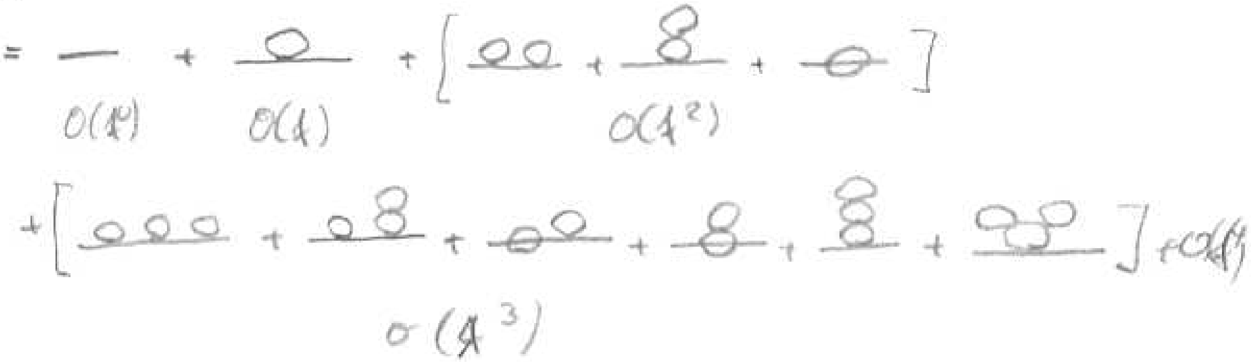
Self energy and vertex function

- We will formulate a relation between the 2-point function and the 1PI Green's function generated by $\Gamma[\varphi]$. This will be useful for the concept of renormalisation

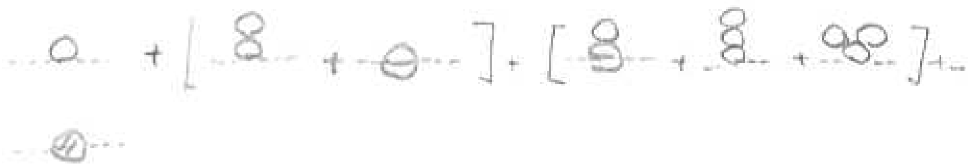
- Ignoring numerical factors we have the following schematic expansion:

$G^{(2)}_{E,c}$ =



- self energy $\Sigma(p)$: going to momentum space and removing external legs

we def $-\Sigma(p)$ =



the sum over all 1PI diagrams to all orders in λ

- Fourier trafo of $G^{(2)}_E$ reads

(recall p 24) $\quad (2\pi)^4\,\delta(p+q)\,\tilde{G}^{(2)}_{E,c}(p,q) \equiv$

to leading order $\quad G^{(2)}_E(x,y)\big|_{O(\lambda^0)} = \langle\phi(x)\phi(y)\rangle_0 = \int \frac{d^4p}{(2\pi)^4}\,\frac{e^{ip(x-y)}}{p^2+m^2}$

$$\Rightarrow\quad (2\pi)^4\,\delta^{(4)}(p+q)\,\tilde{G}^{(2)}_{E,c}(p,q) = \int d^4x\,d^4y\; e^{-ipx}e^{-qy}\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik(x-y)}}{k^2+m^2} = (2\pi)^{2\cdot 4}\int\frac{d^4k}{(2\pi)^4}\,\frac{\delta(k-p)\,\delta(k+q)}{k^2+m^2}$$

$$\Leftrightarrow\quad \boxed{\tilde{G}_E(p,-p)\Big|_{\lambda^0} = \frac{1}{p^2+m^2} = G_0(p)} \qquad , \boxed{G_0^{-1} = p^2+m^2}$$

$$\Rightarrow\quad \tilde{G}_E = G_0 + G_0(-\Sigma)G_0 + G_0(-\Sigma)G_0(-\Sigma)G_0 + \ldots = G_0\,\frac{1}{1+\Sigma G_0}$$

full propagator

$$\Leftrightarrow\quad \boxed{\tilde{G}_E = \frac{1}{G_0^{-1}+\Sigma} = \frac{1}{p^2+m^2+\Sigma}}$$

coming back to $\Gamma[\varphi]$ it holds (ex 7)

$$\circledast \quad \boxed{\int dz\, G_E^{(2)}(x,z)\,\Gamma_E^{(2)}(z,y) = -\delta^{(4)}(x-y)}$$

or in Fourier space $\quad \tilde{G}_E^{(2)}(p,-p)\,\tilde{\Gamma}_E^{(2)}(p,-p) = 1$

$$\Rightarrow \quad \boxed{\tilde{\Gamma}_E^{(2)}(p,-p) = p^2 + m^2 + \Sigma(p)} \quad \text{2 point-vertex function}$$

- $\exists$ higher order relations of the type $\circledast$ between W and Γ [R, ch 7]

3.2 Bare and renormalised Green functions

We had for the full prop. in Fourier space

$$\langle \tilde{\phi}(p)\tilde{\phi}(q)\rangle = \delta(p+q)(2\pi)^4\,\tilde{G}_E = \frac{1}{G_0^{-1}(p) + \Sigma(p)} = \frac{1}{p^2+m^2+\Sigma(p)}$$

Let's compute Σ to $\mathcal{O}(\lambda^1)$ (see p48):

$$\Sigma(p) = \frac{\lambda}{2}\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2+m^2} + \mathcal{O}(\lambda^2) \qquad \left(\text{-O-} = \langle\phi(z)\phi(z)\rangle_0\right)$$

After dimensional regularisation (ex 7) we have

$$d = 4-2\varepsilon \qquad \underline{\Sigma(p) = \frac{\lambda}{2}\left(-\frac{m^2}{16\pi^2}\cdot\frac{1}{\varepsilon} + \mathcal{O}(1)\right)} \qquad \text{(details see also below)}$$

$$\Rightarrow \quad \langle\tilde{\phi}(p)\tilde{\phi}(q)\rangle = (2\pi)^4\delta^{(4)}(p+q)\,\frac{1}{p^2+m^2-\frac{\lambda m^2}{32\pi^2}\left(\frac{1}{\varepsilon}+\mathcal{O}(1)\right) + \mathcal{O}(\lambda^2)}$$

- The pole of the propagator for $p \in \mathbb{C}$ is called

def pole mass, physical mass $\quad m_{pole}^2 = m^2 - \frac{\lambda m^2}{32\pi^2\varepsilon} + \dots$

m^2 finite $\Rightarrow$ m_{pole}^2 infinite OR m^2 infinite $\Rightarrow$ m_{pole}^2 finite

A change of paradigm

From now on we will denote the fields and parameters in the Lagrange density we start with as <u>bare</u> (B) objects:

$$\phi \to \phi_B, \quad m^2 \to m_B^2, \quad \lambda \to \lambda_B$$

$$\Rightarrow \qquad \mathcal{L}_E = \frac{1}{2}\partial_\mu\phi_B\partial_\mu\phi_B + \frac{1}{2}m_B^2\phi_B^2 + \frac{1}{4!}\lambda_B\phi_B^4$$

On the other hands we have measurabe quantities (e.g. particle masses as poles or resonances), these are called <u>renormalised</u> (R) objects. After fixing our conventions they should have a 1 to 1 correspondence to bare objects:

$$\phi_B \equiv Z_\phi^{\frac{1}{2}}\phi_R$$
$$m_B^2 = Z_{m^2} m_R^2$$
$$\lambda_B = Z_\lambda \lambda_R$$

Recall that in our free theory ($\lambda = 0$) we had no divergencies

$$\Rightarrow \quad Z_{i=\phi,m} = 1 + O(\lambda_R) \equiv 1 + \delta Z_i$$

Renormalisability

A theory is renormalisable if there exists a choice of Z_i, such that <u>all</u> renormalised Green's functions and physical quantities remain finite when taking $\varepsilon \to 0$.

[proof for ϕ^4-theory: e.g. Ryder, chap. 9.3 & refs] it is essential that the "counterterms" introduced to remove the divergencies are of the same form as the original Lagrangian

Let us return to our example:

$$\langle \tilde{\phi}_B(P)\,\tilde{\phi}_B(Q)\rangle = \delta^{(4)}(P+Q)(2\pi)^4 \, \frac{1}{P^2+m_B^2-\frac{\lambda_B}{2}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2+m_B^2}+O(\lambda_B)}$$

insert
$$m_B^2 = m_R^2 + \delta Z_{m^2} m_R^2 + O(\lambda_R^2)$$
$$\lambda_B = \lambda_R + \underbrace{\delta Z_\lambda \lambda_R}_{O(\lambda_R^2)}$$

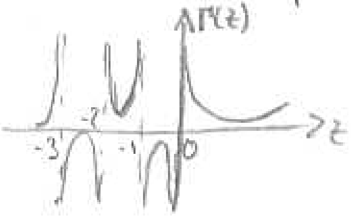
$$\Rightarrow m_{pole}^2 = m_R^2 + \delta Z_{m^2} m_R^2 - \frac{\lambda_R}{2}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2+m_R^2} + O(\lambda_R^2)$$

& 7:
$$\boxed{\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2+m_R^2} = \frac{\Gamma\left(1-\frac{d}{2}\right)}{(4\pi)^{\frac{d}{2}}\,(m_R^2)^{1-\frac{d}{2}}} \quad \overset{d=4-2\varepsilon}{=} \quad \frac{m_R^2}{(4\pi)^2}(4\pi)^{\varepsilon}(m_R^2)^{-\varepsilon}\,\Gamma(-1+\varepsilon)}$$

- The Gamma function

$$\Gamma(z) = \int_0^\infty dt\, t^{z-1} e^{-t}, \qquad \Gamma(n+1) = n! \text{ for } n\in\mathbb{N}$$

it has simple poles at $z=-n$, $n\in\mathbb{N}$ (with residue $\frac{(-1)^n}{n!}$)



$$\Gamma(z+1) = z\,\Gamma(z)$$

Euler-Mascheroni constant 0,577...

$$\Rightarrow \Gamma(-1+\varepsilon) = \frac{1}{-1+\varepsilon}\Gamma(\varepsilon) = \frac{1}{(\varepsilon-1)\varepsilon}\underbrace{\Gamma(1+\varepsilon)}_{1-\gamma_{Euler}\varepsilon+O(\varepsilon^2)}$$

- we introduce a mass scale μ to write

$$(m_R^2)^{-\varepsilon} = \mu^{-2\varepsilon}\left(\frac{\mu^2}{m_R^2}\right)^{\varepsilon} = \mu^{-2\varepsilon}\left(1+\varepsilon\ln\frac{\mu^2}{m_R^2}+O(\varepsilon^2)\right)$$ as $a^\varepsilon = e^{\varepsilon \ln a} = 1+\varepsilon\ln a + O(\varepsilon^2)$

$$\Rightarrow m_{Pole}^2 = m_R^2 + \delta Z_{m^2} m_R^2 + \frac{\lambda_R m_R^2 \mu^{-2\varepsilon}}{32\pi^2}\left(\frac{1}{\varepsilon}+1-\gamma_E+\ln\frac{\mu^2}{m_R^2}+\ln 4\pi + O(\varepsilon)\right)$$

minimal subtraction (MS scheme)

$$\delta Z_{m^2} = -\frac{\lambda_R\mu^{-2\varepsilon}}{32\pi^2}\cdot\frac{1}{\varepsilon} + O(1)$$ makes m_{pole}^2 finite

- the subtraction scheme is not unique (there are others)

- Literature conventions: often $\lambda_R \mu^{-2\varepsilon} \to \lambda_R$

$$-\gamma_E + \ln\mu^2 + \ln 4\pi = \ln\left(\frac{4\pi\mu^2}{e^{\gamma_E}}\right) \equiv \ln\bar{\mu}^2$$

Summary $\quad m_B^2 \xleftrightarrow{Z_m^2} m_R^2 \xleftrightarrow{1+O(\lambda_R)} m_{pole}^2$

bare,	renorm.,	finite and
∞	finite,	unique (physical mass
	(not unique)	

<u>Renormalised Green's functions</u>

$$G^{(n)}_{B,c}(x_1 \ldots x_n) = \langle \phi_B(x_1) \ldots \phi_B(x_n) \rangle_c \underset{\uparrow}{=} Z_\phi^{\frac{n}{2}} G^{(n)}_{R,c}$$

$$G^{(n)}_{R,c}(x_1 \ldots x_n) = \langle \phi_R(x_1) \ldots \phi_R(x_n) \rangle_c \qquad , \ \phi_B = Z_\phi^{\frac{1}{2}} \phi_R$$

- the same holds after Fourier trafo

$$\Rightarrow \tilde{A}^{(n)}_{B,c} = \left[\tilde{G}^{(2)}_{B,c}(p_1, -p_1)\right]^{-1} \cdots \tilde{G}^{(n)}_{B,c} = Z_\phi^{-\frac{n}{2}} \tilde{A}^{(n)}_{R,c}$$

$\to$ the amplitudes containing $Z_\phi^{\frac{n}{2}} \tilde{A}^{(n)}_{B,c}$ after LSZ reduction are finite.