3.3. Renormalisation Group (RG)

- in order to regularise the 2-point function we have introduced a mass scale "μ". In general the renormalisation factors Z_i will depend on μ, as we have seen for Z_{m^2} (and will see for Z_λ below). Also m_R, λ_R will in general on μ (unlike m_B, λ_B).
But $G_B^{(n)}$ is indep of μ => the (dimensionless) derivative

$$\mu \frac{d}{d\mu} G_B^{(n)}(x_1 \ldots x_n; m_B, \lambda_B) = 0$$

where we made the parameter-dependence explicit

From $G_B^{(n)}(x_1, \ldots, x_n; m_B, \lambda_B) = Z_\phi^{\frac{n}{2}} G_R^{(n)}(x_1, \ldots, x_n; m_R, \lambda_R)$ it follows

$$0 = \mu \frac{d}{d\mu}\left(Z_\phi^{\frac{n}{2}} G_R^{(n)}\right) = \frac{n}{2}\mu \frac{\partial Z_\phi}{\partial \mu} Z_\phi^{\frac{n}{2}-1} G_R^{(n)} + Z_\phi^{\frac{n}{2}}\left(\mu \frac{\partial}{\partial\mu} G_R^{(n)} + \mu \frac{\partial m_R}{\partial \mu}\frac{\partial}{\partial m_R} G_R^{(n)} + \mu \frac{\partial \lambda_R}{\partial \mu}\frac{\partial G_R^{(n)}}{\partial \lambda_R}\right)$$

<=>

$$\boxed{\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda_R)\frac{\partial}{\partial \lambda_R} + m\gamma_m(\lambda_R)\frac{\partial}{\partial m_R} + n\gamma(\lambda_R)\right] G_R^{(n)} = 0}$$

RG eq. for MS

with $\beta(\lambda_R) = \mu \frac{\partial \lambda_R}{\partial \mu}$, $m\gamma_m(\lambda_R) = \mu \frac{\partial m_R}{\partial \mu}$, $\gamma(\lambda_R) = \mu \frac{\partial}{\partial \mu} \ln \sqrt{Z_\phi}$

The study of the β-function $\beta(\lambda_R)$ will be important later.

- these eqs can be formulated in Fourier space and/or on $\Gamma^{(n)}$ instead.

- the RG eqs. can be formulated differently:

Callan-Symanzik: $m_R \frac{\partial}{\partial m_R} G = \ldots$ involving only R-quantities

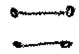
Wilson RG: $\Lambda \frac{\partial}{\partial \Lambda} G = \ldots$ in terms of a cutoff Λ (here IR!)

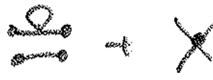
(see also Franz Wegner)

- RG-studies used also in phase-transitions in stat. Phys!

Regularisation of λ and determination of Z_λ

Therefore we will consider $G^{(4)}_{B,c}$ to order λ^2: from page 37 or using the path integral formalism we have

$G^{(4)}_B$ $\quad \underline{\mathcal{O}(\lambda_B^0)}$: $\langle \phi_1 \dots \phi_4 \rangle_0$ or  only disconnected

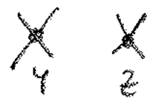
$\underline{\mathcal{O}(\lambda_B^1)}$: $\langle \phi_1 \dots \phi_4 \int \phi^4 \rangle_0 - \langle \phi_1 \dots \phi_4 \rangle_0 \langle \int \phi^4 \rangle$ or 

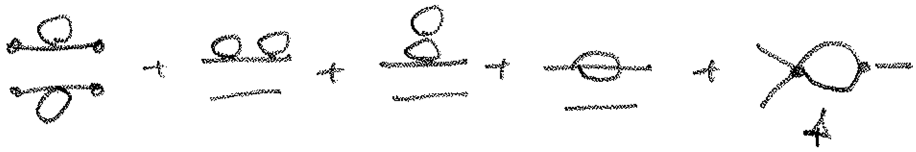
$\Rightarrow$ only  contracted with  ; disconn, connected

we get for $\underline{\mathcal{O}(\lambda_B^2)}$:

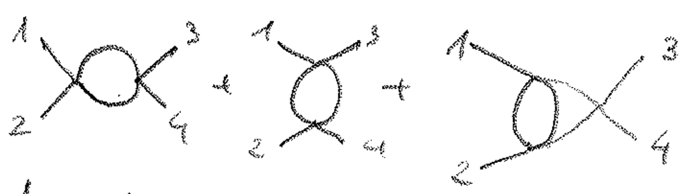
$$\tfrac{1}{2} \langle \phi_1 \dots \phi_4 \int \phi^4 \int \phi^4 \rangle_0 - 2 \langle \phi_1 \dots \phi_4 \int \phi^4 \rangle_0 \langle \int \phi^4 \rangle_0$$

$$- \tfrac{1}{2} \langle \phi_1 \dots \phi_4 \rangle_0 \langle \int \phi^4 \int \phi^4 \rangle_0 + \langle \phi_1 \dots \phi_4 \rangle_0 \langle \int \phi^4 \rangle_0^2$$

$\Rightarrow$ only terms connecting  with both 

that is 

only connected

[in fact we have 3 terms 

$\Rightarrow$ Mandelstam variables for external momenta $p_{1 \dots 4}$

$$s = (p_1 + p_2)^2, \quad t = (p_1 + p_3)^2, \quad u = (p_1 + p_4)^2$$]

- for the explicit computation we go to Fourier space

$$(2\pi)^4 \delta^{(4)}(p_1 + p_2 + p_3 + p_4)\, \tilde{G}_{Bc}(p_1 \dots p_4) = \prod_{i=1}^{4} \int dx_i\, e^{i p_i x_i}\, G_B^{(4)}(x_1 \dots x_4)$$

and set the external momenta $p_1 = \dots = p_4 = 0$ (this is all we need)

- for $p_i \neq 0$ one can use the Feynman parametrisation $\to$ ext.

$\underline{O(\lambda_B^1)}$: $(2\pi)^4\delta(0)\, \tilde{G}^{(4)}_{B,c}(0,\dots,0)\Big|_{\lambda_B} = \int_{x_1..x_4} e^{i p_j x_j} \int_w \left(-\frac{\lambda_B}{4!}\right) \langle \phi_B(x_1)\phi_B(x_2)\phi_B(x_3)\phi_B(x_4)\,\phi_B^4(w)\rangle_0$

$$= -\lambda_B \underbrace{\int_{x_1..4}}_{\prod_{i=1}^{4}\delta(p_i)} \underbrace{\int_w}_{\delta(p_1+p_2+p_3+p_4)} \int_{p_1..4} \frac{e^{ip_1(x_1-w)}}{p_1^2+m_B^2}\cdot\frac{e^{ip_2(x_2-w)}}{p_2^2+m_B^2}\cdot\frac{e^{ip_3(x_3-w)}}{p_3^2+m^2}\cdot\frac{e^{ip_4(x_4-w)}}{p_4^2+m^2}$$

$$= -\lambda_B\,\delta(0)\,\frac{1}{(m_B^2)^4} \qquad \text{to LO} \quad \times \quad \sim \lambda_B$$

$\underline{O(\lambda_B^2)}$:

$$(2\pi)^4\delta(0)\, G^{(4)}_{B,c}(0,\dots,0)\Big|_{\lambda_B^2} = \int_{x_1..4}\int_{w,v}\frac{1}{2}\frac{\lambda_B^2}{(4!)^2}\langle \underbrace{\phi_B(x_1)\phi_B(x_2)\phi_B(x_3)\phi_B(x_4)}_{8}\,\phi_B(w)^4\,\phi_B(v)^4\rangle_0 \quad \text{etc}$$

Connected, only contr. with w and v

$$= \dots = \frac{3}{2}\lambda_B^2\,\delta(0)\,\frac{1}{(m_B^2)^4}\underbrace{\int\frac{d^4p}{(2\pi)^4}\frac{1}{(p^2+m_B^2)^2}}_{\text{divergent, regularise}}$$

in $d=4-2\epsilon$
$$\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2+m^2} = -\frac{\mu^{-2\epsilon}m^2}{(4\pi)^2}\left(\frac{1}{\epsilon}+1\underbrace{-\gamma_E+\ln\frac{\mu^2}{m^2}+\ln(4\pi)}_{\ln\frac{\bar{\mu}^2}{m^2}}+O(\epsilon)\right)$$

$\Rightarrow$
$$\int\frac{d^dp}{(2\pi)^d}\frac{1}{(p^2+m^2)^2} = -\frac{d}{dm^2}\int\frac{d^dp}{(2\pi)^d}\frac{1}{p^2+m^2} = -\frac{\mu^{-2\epsilon}}{(4\pi)^2}\left(\frac{1}{\epsilon}+\ln\frac{\bar{\mu}^2}{m^2}+O(\epsilon)\right)$$

- using $\lambda_B = (1+\delta Z_\lambda)\lambda_R$ (and the fact that $Z_\phi = 1+O(\lambda_R^2)$ so that it can be neglected here in $G_B^{(4)} = Z_\phi^2 G_R$)

we find

$$\underbrace{(m_B^2)^4\,\delta(0)\,\tilde{G}_{R,c}(0\dots0)}_{\text{removing ext. legs}} = \lambda_{phys} = \lambda_R + \lambda_R\,\delta Z_\lambda - \frac{3}{2}\frac{\lambda_R^2\mu^{-2\epsilon}}{(4\pi)^2}\left(\frac{1}{\epsilon}+\ln\frac{\bar{\mu}^2}{m_R^2}+O(\epsilon)\right) + O(\lambda_R^3)$$

- in the term of order λ_B^2 we can replace $B \to R$ everywhere to L.O.

$\Rightarrow$ in the MS scheme we chose $\boxed{\delta Z_\lambda = \frac{3}{2}\,\frac{\lambda_R\,\mu^{-2\varepsilon}}{(4\pi)^2}\cdot\frac{1}{\varepsilon}}$

Determination of $\beta(\lambda) = \mu\frac{\partial\lambda_R}{\partial\mu}$: (in L.O. in pert. theory)

$$\lambda_B = \lambda_R Z_\lambda = \lambda_R + \frac{3}{2}\frac{\lambda_R^2\,\mu^{-2\varepsilon}}{(4\pi)^2}\frac{1}{\varepsilon} + O(\lambda_R^3)$$

$$\Rightarrow 0 = \mu\frac{d}{d\mu}\lambda_B = \mu\frac{\partial\lambda_R}{\partial\mu} + \frac{3}{2}\frac{\lambda_R^2}{(4\pi)^2}\underbrace{\mu\frac{\partial}{\partial\mu}\frac{\mu^{-2\varepsilon}}{\varepsilon}}_{-2\mu^{-2\varepsilon}} + \frac{3}{2}\frac{\mu^{-2\varepsilon}}{(4\pi)^2\varepsilon}\,2\lambda_R\underbrace{\mu\frac{\partial\lambda_R}{\partial\mu}} + O(\lambda_R^3)$$

if we reiterate the eq. this term is of $O(\lambda_R^3)$!

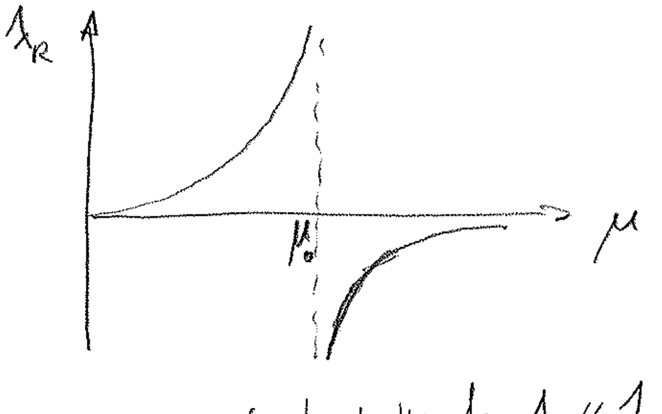
$\Rightarrow$ to L.O. $\boxed{\mu\frac{\partial\lambda_R}{\partial\mu} = \frac{3}{(4\pi)^2}\lambda_R^2}$

- the derivation of such RG equations is simple once the Z_i - factors are known
- the eqs are finite in the limit $\varepsilon \to 0$ as these are eqs. for renormalised quantities :

Solution (ex 7.4) for $\lambda_R(\mu)$ to this order:

$$\lambda_R(\mu) = \frac{(4\pi)^2}{3}\cdot\frac{1}{\ln\frac{\mu_0}{\mu}}$$

with some (dim-full) constant μ_0.

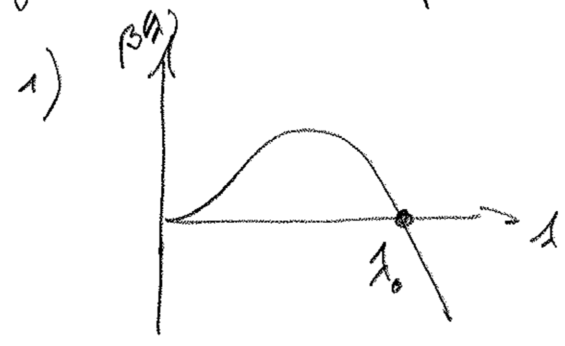


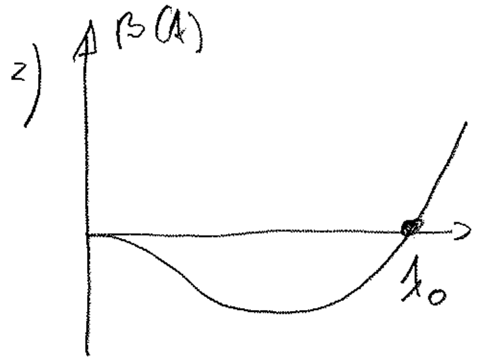
we can only trust this for $\lambda_R \ll 1$ (and $\mu \ll \mu_0$)

Fixed points of the β-function

We had defined $\underline{\beta(\lambda_R) = \mu \frac{\partial \lambda_R}{\partial \mu}}$. Let us discuss 2

generic scenarios for $\beta(\lambda)$:





(One could hop that higher orders remove the pole in $\lambda_R(\mu)$ and map ϕ^4-theory to scenario 1) – this seems only be true for $d = 3$)

- λ_0 is a fixed point (FP) of $\beta(\lambda)$ – is it stable?

around $\lambda \approx \lambda_0$ we can linearise:

$$\beta(\lambda) \approx -B(\lambda - \lambda_0) \qquad \text{with} \quad B > 0 \quad \text{in } 1)$$
$$= \mu \frac{\partial \lambda}{\partial \mu} \qquad\qquad\qquad B < 0 \quad \text{in } 2)$$

$\Rightarrow$ solution $\underline{\lambda(\mu) = \lambda_0 + e^{-B \ln \mu}} = \underline{\lambda_0 + \frac{1}{\mu^B}}$

$\Rightarrow$ the scale dependent (running) coupling $\lambda(\mu)$ is driven towards λ_0 for

$$B > 0 \quad \lim_{\mu \to \infty} \left(\lambda(\mu) = \lambda_0 + \frac{1}{\mu^B}\right) = \lambda_0 \quad 1)$$

$$B < 0 \quad \lim_{\mu \to 0} \left(\lambda(\mu) = \lambda_0 + \mu^{|B|}\right) = \lambda_0 \quad 2)$$

Therefore in 1) we have a UV FP

2) we have an IR FP – this is possible in an asymptot. free theory