4. Fermions

4. Quantisation of the Dirac field

(back to Minkowski space)

- We have seen $(\partial_\mu \partial^\mu + m^2)\phi(x^\mu) = \underline{(\partial_0^2 - \vec{\nabla}^2 + m^2)\phi(x^0, \vec{x}) = 0}$ KG eq (2. order rel. diff. eq)

can be solved with plane waves:

$\underline{\phi(x^\mu) = N(p)\, e^{-ip_\mu x^\mu}} \overset{KG}{\Rightarrow} (-p^{0\,2} + \vec{p}^2 + m^2) N(p) e^{-ip_\mu x^\mu} = 0$

with $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$ sol has $p_0 = \pm E_{\vec{p}}$ pos/neg energy

- to quantise we wrote

$\phi \to \hat{\phi}(x^\mu) = \int d^3\vec{p}\, [\hat{N}_+(p)\, e^{-iE_{\vec{p}}x^0 + i\vec{p}\vec{x}} + \hat{N}_-(p)\, e^{+iE_{\vec{p}}x^0 + i\vec{p}\vec{x}}]$ Superpos. of pos. & neg energy

$\Rightarrow$ canon. commut. relations for $\pi \to \hat{\pi}$, $H \to \hat{H}$, $\hat{\phi}$ yielded $\hat{N}_{\pm} \sim \hat{a}^{(\dagger)}$ with HO commut. relations, for single scalar spin 0 field

- Here starting point:

$$\boxed{0 = (i\gamma^\mu \partial_\mu - m)\Psi} = (i\cancel{\partial} - m)\Psi = \begin{pmatrix} -m & i\partial_0 + \vec{\sigma}\vec{\nabla} \\ i(\partial_0 - \vec{\sigma}\vec{\nabla}) & -m \end{pmatrix}\Psi$$

, $\vec{\sigma}$ Pauli matrices

<u>Dirac equation</u>, "simplest" 1. order relativistic (Lorentz inv) diff. eq.

- same procedure: construct classical solutions (plane waves), Hamiltonian
  $\to$ quantise field Ψ (complex 4vector, spinor; spin $\frac{1}{2}$ particle) as superposition
  $\to$ canon. quant. rules yield commut. relations for creation/annihil. operators.

Notation: $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$, $\sigma^\mu = (1, \vec{\sigma})$, $\bar{\sigma}^\mu = (1, -\vec{\sigma})$

$[\sigma^i, \sigma^j] = 2i\varepsilon^{ijk}\sigma^k$ Pauli matrices $\Rightarrow \{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$

$(\sigma^j)^2 = \mathbb{1}$ anti-com Clifford alg.

note $\sigma^{j} = \sigma_{j}$

$\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$

- Lagrange density $\boxed{\mathcal{L}_M = \bar\Psi (i\gamma^\mu\partial_\mu - m)\Psi}$, $\bar\Psi \equiv \Psi^\dagger\gamma^0$

$\Rightarrow$ e.o.m. $\frac{\partial\mathcal{L}}{\partial\Psi} - \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)} = 0 \Leftrightarrow -i\,\partial_\mu\bar\Psi\gamma^\mu - m\bar\Psi = 0 \overset{+\gamma_0}{\Leftrightarrow}$ Dirac eq.

- Hamiltonian (of several fields φ_i, here $\Psi, \bar\Psi$)

$$\mathcal{H} = \sum_i \pi_i\,\partial_0\varphi_i - \mathcal{L}_M(\{\varphi_i\})$$, here $\pi = \frac{\partial\mathcal{L}_M}{\partial(\partial_0\Psi)} = i\bar\Psi\gamma^0$

$\bar\pi = \frac{\partial\mathcal{L}_M}{\partial(\partial_0\bar\Psi)} = 0$

$\Rightarrow \mathcal{H} = i\bar\Psi\gamma^0\partial_0\Psi - \{\bar\Psi i\gamma^\mu\partial_\mu\Psi - \bar\Psi m\Psi\}$

$$\boxed{\mathcal{H} = \bar\Psi[-i\gamma^j\partial_j + m]\Psi}$$ (j-summation 1,2,3)

classical solutions of the Dirac eq:

- note that $(-i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu - m) = (\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)$ (commute; $\rightarrow \frac{1}{2}\{\gamma^\mu,\gamma^\nu\} = \eta^{\mu\nu}$)

$= (\partial^\mu\partial_\mu + m^2)$

$\Rightarrow$ if Ψ solves Dirac it also solves KG

$\Rightarrow$ an ansatz $\Psi(x^\mu) = u(p)\,e^{-ip_\mu x^\mu}$ (u: 4 vector) has to satisfy $p_0 = \pm E_p = \pm\sqrt{\vec p^{\,2} + m^2}$

$\rightarrow$ take lin. comb. of pos. and neg. energy solutions.

and $(\gamma^\mu p_\mu - m)\,u(p) = 0$

- determination of $u(p)$: choose rest frame first ($m \neq 0$!): $p = p_0 = (m, \vec 0)$

$$(m\gamma^0 - m)\,u(p_0) = m\begin{pmatrix}-1 & 1\\ 1 & -1\end{pmatrix}u(p_0) \Rightarrow u(p_0) = \sqrt{m}\begin{pmatrix}\xi\\ \xi\end{pmatrix}$$, ξ 2 vector

$\Rightarrow \exists$ 2 indep solutions ξ^s, $s = 1,2$, orthonormalize them

Lorentz boost $\Rightarrow u(p;s) = \begin{pmatrix}\sqrt{p_\mu\sigma^\mu}\,\xi^s\\ \sqrt{p_\mu\bar\sigma^\mu}\,\xi^s\end{pmatrix}$ $s = 1,2$ see ch 3.3 [PS]

ditto for neg energies $\Psi = v(p)\,e^{+ip_0x^0 + i\vec p\vec x}$ $\rightarrow v(p,s) = \begin{pmatrix}\sqrt{p_\mu\sigma^\mu}\,\eta^s\\ -\sqrt{p_\mu\bar\sigma^\mu}\,\eta^s\end{pmatrix}$, $s = 1,2$

$\Rightarrow (\gamma^\mu p_\mu + m)\,v(p) = 0$

defining $\bar{u} = u^\dagger \gamma^0$, $\bar{v} = v^\dagger \gamma^0$ we get

- normalisation: $u^\dagger(p;s)\, u(p;s') = 2E_{\vec p}\, \delta_{ss'}$, $v^\dagger(p;s)\, v(p;s') = 2E_{\vec p}\, \delta_{ss'}$

- orthogonality: $\bar{u}(p;r)\, v(p;s) = 0 = \bar{v}(p;r)\, u(p;s)$
  $u^\dagger(p^0;\vec p;r)\, v(p^0;-\vec p;s) = 0 = v^\dagger(p^0;-\vec p;r)\, u(p^0;\vec p;s)$

- completeness:

$$\sum_{s=1,2} v(p;s)\,\bar{v}(p;s) = \gamma^\mu p_\mu - m \qquad \underset{\text{(scalar)}}{\sum_{s=1,2}} u(p;s)\,\bar{u}(p;s) = \gamma^\mu p_\mu + m$$

- Quantisation: ψ superposition with 1 operator per solution

$$\psi \to \hat\psi = \int \frac{d^3\vec p}{\sqrt{(2\pi)^3\, 2E_{\vec p}}} \sum_{s=1,2} \left[\hat a^{(s)}_{\vec p}\, u(\vec p;s)\, e^{-ip^\mu x_\mu} + \hat b^{\dagger(s)}_{\vec p}\, v(\vec p;s)\, e^{ipx} \right]$$

$$\hat{\bar\psi} = \int_{\vec p} \sum_s \left[\hat a^{\dagger(s)}_{\vec p}\, \bar u\, e^{ip\cdot x} \quad \hat b^{(s)}_{\vec p}\, \bar v \quad e^{-ip\cdot x} \right]$$

in $\hat{\mathcal{H}}$

$$\Rightarrow [-i\gamma^j \partial_j + m]\,\hat\psi = \int_{\vec q} \sum_t \Big[\hat a^{(t)}_{\vec q} \underbrace{(-\gamma^j q_j + m)\, u(\vec q;t)}_{= E_{\vec q}\gamma_0 u} e^{-iq\cdot x} + \hat b^{\dagger(t)}_{\vec q} \underbrace{(\gamma^j q_j + m)\, v(\vec q;t)}_{-E_{\vec q}\gamma_0 v} e^{iq\cdot x} \Big]$$

$$\Rightarrow \hat H = \int d^3\vec x\; \hat{\bar\psi}(x) [-i\gamma^j \partial_j + m]\, \psi(x)$$

$$= \int_{\vec x}\int_{\vec p}\int_{\vec q} \sum_{s,t} \left[\hat a^{\dagger(s)}_{\vec p}\, \bar u(p;s)\, e^{ipx} + \hat b^{(s)}_{\vec p}\, \bar v(p;s)\, e^{-ip\cdot x} \right] E_{\vec q}\gamma_0 \left[\hat a^{(t)}_{\vec q}\, u(q;t)\, e^{-iqx} - \hat b^{\dagger(t)}_{\vec q}\, v(q;t)\, e^{iqx} \right]$$

$$= \int_{\vec x}\int_{\vec p}\int_{\vec q} \sum_{s,t} E_{\vec q} \Big\{ \bar u(p;s)\gamma_0 u(q;t)\, \hat a^{\dagger(s)}_{\vec p} \hat a^{(t)}_{\vec q}\, e^{i(E_{\vec p}-E_{\vec q})x_0 - i(\vec p-\vec q)\vec x} - \bar u(p;s)\gamma_0 v(q;t)\, \hat a^{\dagger(s)}_{\vec p} \hat b^{\dagger(t)}_{\vec q}\, e^{i(E_{\vec p}+E_{\vec q})x_0 - i(\vec p+\vec q)\vec x}$$

$$+ \bar v(p;s)\gamma_0 u(q;t)\, \hat b^{(s)}_{\vec p} \hat a^{(t)}_{\vec q}\, e^{i(E_{\vec p}+E_{\vec q})x_0 - i(\vec p+\vec q)\vec x} - \bar v(p;s)\gamma_0 v(q;t)\, \hat b^{(s)}_{\vec p} \hat b^{\dagger(t)}_{\vec q}\, e^{-i(E_{\vec p}-E_{\vec q})x_0 + i(\vec p-\vec q)\vec x} \Big\}$$

use that $\int_x$ gives $\delta^{(3)}(\vec p \mp \vec q)$

$$\Rightarrow \hat{H} = \int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} E_{\vec{p}} (2\pi)^3 \sum_{s,t} \Big\{ u^\dagger(p,s) u(p,t) \hat{a}^{\dagger(s)}_{\vec{p}} \hat{a}^{(t)}_{\vec{p}} - u^\dagger(p,s) v(p^0,-\vec{p},t) \hat{a}^{(s)}_{\vec{p}} \hat{b}^{(t)}_{-\vec{p}} \cdot e^{2iE_{\vec{p}}x^0}$$

$$+ v^\dagger(p,s) u(p,t) \hat{b}^{(s)}_{p} \hat{a}^{(t)}_{-\vec{p}} e^{2iE_{\vec{p}}x_0} - v^\dagger(p,s) v(p,t) \hat{b}^{(s)}_{\vec{p}} \hat{b}^{\dagger(t)}_{\vec{p}} \Big\}$$

$$\hat{H}^{\text{normal}} = \int d^3\vec{p}\, E_{\vec{p}} \sum_{s=1,2} \left\{ \hat{a}^{\dagger(s)}_{\vec{p}} \hat{a}^{(s)}_{\vec{p}} - \hat{b}^{(s)}_{\vec{p}} \hat{b}^{\dagger(s)}_{\vec{p}} \right\}$$

- we can either im commut relations on $\hat{\psi}, \hat{\psi}^\dagger \Rightarrow$ comm. rel on $\hat{a}$'s, $\hat{b}$'s

OR **impose** comm. rel on $\hat{a}$'s & $\hat{b}$'s $\Rightarrow$ CR on ψ's

* for annihilation op of positive energy states we expect to lower the energy $[\hat{H}, \hat{a}_{\vec{p}}] = \hat{H}\hat{a}_{\vec{p}} - \hat{a}_{\vec{p}}\hat{H} = -E_{\vec{p}}\hat{a}_{\vec{p}}$ until we reach $|0\rangle$

$$\Leftrightarrow \quad \hat{H}\hat{a}_{\vec{p}} = \hat{a}_{\vec{p}}(\hat{H} - E_{\vec{p}})$$

and for creation op to raise the energy

$$[\hat{H}, \hat{a}^\dagger_{\vec{p}}] = +E_{\vec{p}}\hat{a}^\dagger_{\vec{p}}$$

- <u>1st attempt of quantisation:</u> assume we have C.R. for $\hat{a}$'s and $\hat{b}$'s as for bosons:

$$[\hat{a}^{(s)}_{\vec{p}}, \hat{a}^{\dagger(t)}_{\vec{q}}] = \delta^{(3)}(\vec{p}-\vec{q})\,\delta_{s,t} = [\hat{b}^{(s)}_{\vec{p}}, \hat{b}^{\dagger(t)}_{\vec{q}}] \quad \text{and the rest } 0$$

$\Rightarrow$ using $[AB,C] = A[B,C] + [A,C]B$ we get

$$[\hat{H}, \hat{a}^{(s)}_{\vec{k}}] = \int d^3\vec{p}\, E_{\vec{p}} \sum_t [\hat{a}^{\dagger(t)}_{p}, \hat{a}^{(s)}_{\vec{k}}] \hat{a}^{(t)}_{\vec{p}} = -E_{\vec{k}}\hat{a}^{(s)}_{\vec{k}} \quad \checkmark$$

$$[\hat{H}, \hat{a}^{(s)\dagger}_{\vec{k}}] = \qquad \hat{a}^{\dagger(t)}_{p} [\hat{a}^{(t)}_{\vec{p}}, \hat{a}^{\dagger(s)}_{\vec{k}}] = +E_{\vec{k}}\hat{a}^{\dagger(s)}_{\vec{k}} \quad \checkmark$$

BUT
$$[\hat{H}, \hat{b}^{(s)}_{\vec{k}}] = \int d^3\vec{p}\, E_{\vec{p}} \sum_t -\hat{b}^{(t)}_{\vec{p}} [\hat{b}^{(t)\dagger}_{\vec{p}}, \hat{b}^{(s)}_{\vec{k}}] = +E_{\vec{k}}\hat{b}^{(s)}_{\vec{k}} \quad ↯$$

$$[\hat{H}, \hat{b}^{(s)\dagger}_{\vec{k}}] = \qquad -[\hat{b}^{(t)}_{\vec{p}}, \hat{b}^{(s)\dagger}_{\vec{k}}] \hat{b}^{(t)\dagger}_{\vec{p}} = -E_{\vec{k}}\hat{b}^{\dagger(s)}_{\vec{k}} \quad ↯$$

- 2nd attempt: use anti-commutators instead!

$$\{\hat{a}^{(s)}_{\vec{p}}, \hat{a}^{(t)\dagger}_{\vec{q}}\} = \delta^{(3)}(\vec{p}-\vec{q})\,\delta_{st} = \{\hat{b}^{(s)}_{\vec{p}}, \hat{b}^{(t)\dagger}_{\vec{q}}\}$$

and $\{\hat{a},\hat{a}\} = 0 = \{\hat{a}^\dagger,\hat{a}^\dagger\}$, $\{\hat{b},\hat{b}\} = 0 = \{\hat{b}^\dagger,\hat{b}^\dagger\}$, $\{\hat{a},\hat{b}\} = \{\hat{a}^\dagger,\hat{b}\} = \{\hat{a}^\dagger,\hat{b}^\dagger\} = 0$

$\Rightarrow$ use $[AB,C] = A\{B,C\} - \{A,C\}B$ to get

$$[\hat{H}, \hat{a}^{(s)}_{\vec{k}}] = \int d^3\vec{p}\, E_{\vec{p}} \sum_t -\{\hat{a}^{(t)\dagger}_{\vec{p}}, \hat{a}^{(s)}_{\vec{k}}\}\hat{a}^{(t)}_{\vec{p}} = -E_{\vec{k}}\hat{a}^{(s)}_{\vec{k}} \quad \checkmark$$

unchanged

$$[\hat{H}, \hat{a}^{\dagger(s)}_{\vec{k}}] = \qquad + \hat{a}^{\dagger(t)}_{\vec{p}}\{\hat{a}^{(t)}_{\vec{p}}, \hat{a}^{\dagger(s)}_{\vec{k}}\} = +E_{\vec{k}}\hat{a}^{\dagger(s)}_{\vec{k}} \quad \checkmark$$

New

$$[\hat{H}, \hat{b}^{(s)}_{\vec{k}}] = \int d^3\vec{p}\, E_{\vec{p}} \sum_t -\hat{b}^{(t)}_{\vec{p}}\{\hat{b}^{\dagger(t)}_{\vec{p}}, \hat{b}^{(s)}_{\vec{k}}\} = -E_{\vec{k}}\hat{b}^{(s)}_{\vec{k}} \quad \checkmark$$

$$[\hat{H}, \hat{b}^{\dagger(s)}_{\vec{k}}] = \qquad -\left(-\{\hat{b}^{(t)}_{\vec{p}}, \hat{b}^{(s)\dagger}_{\vec{k}}\}\right)\hat{b}^{\dagger(t)}_{\vec{p}} = +E_{\vec{k}}\hat{b}^{\dagger(s)}_{\vec{k}} \quad \checkmark$$

$\Rightarrow$ we need to use anti-com. rel.

Consequences for field operators (ex 8.):

$$\{\hat{\psi}(x^0,\vec{x}), \hat{\psi}(x^0,\vec{y})\} = 0 = \{\hat{\psi}^\dagger(x^0,\vec{x}), \hat{\psi}^\dagger(x^0,\vec{y})\}$$

and

$$\{\hat{\psi}_\alpha(x^0,\vec{x}), \hat{\pi}_\beta(x^0,\vec{y})\} = \{\hat{\psi}_\alpha(x^0,\vec{x}), i\hat{\psi}^\dagger_\beta(x^0,\vec{y})\}$$

$$= i\,\delta^{(3)}(\vec{x}-\vec{y})\,\delta_{\alpha\beta}$$

as in canonical quantisation but with $\{,\}$ instead of $[,]$