4.2. Fermionic Path integrals

This time we will directly switch to the Euclidean path integral formalism. The starting point will be again the n-point Green's functions

$$G_E^{(n)}(x_1,\dots,x_m;x_{m+1},\dots,x_n) \equiv \langle 0|T\{\hat\psi_H(x_1)\dots\hat\psi_H(x_m)\hat{\bar\psi}_H(x_{m+1})\dots\hat{\bar\psi}_H(x_n)\}|0\rangle$$

in the Heisenberg picture. Because of the anti-commutation we have to modify time ordering as follows

$$T\{\hat\psi_H(\tilde x),\hat\psi_H(\tilde y)\} \equiv \Theta(\tilde x^0-\tilde y^0)\hat\psi_H(\tilde x)\hat\psi_H(\tilde y) - \Theta(\tilde y^0-\tilde x^0)\hat\psi_H(\tilde y)\hat\psi_H(\tilde x)$$

- Examples for interacting theories for Fermions are the Gross-Neveu model $\mathcal{L}_M + \lambda(\bar\psi\psi)^2$ or Thirring model $\mathcal{L}_M + \lambda(\bar\psi\gamma^\mu\psi)(\bar\psi\gamma_\mu\psi)$

- In analogy to the scalar field we consider

$$\boxed{G_\beta^{(n)}(x_1,\dots,x_m,\dots x_n) = \frac{\mathrm{Tr}\left[e^{-\beta\hat H}T\{\hat\psi_H(x_1)\dots\hat\psi_H(x_m)\hat{\bar\psi}_H(x_{m+1})\dots\hat{\bar\psi}_H(x_n)\}\right]}{\mathrm{Tr}\, e^{-\beta\hat H}}}$$

for $\beta \geq x_1^0 \dots x_n^0 \geq 0$ $\qquad \xrightarrow{\beta\to\infty} G_E^{(n)}$

- For simplicity we consider again 1dim QM

$$x^\mu \to t,\quad \hat\psi(x^\mu) \to \hat x(t),\quad \hat\pi = i\hat\psi^\dagger \to \hat p(t),\quad \hat H \to \hat H(\hat x,\hat p)$$

the "fermionic oscillator" is def by

$$\boxed{\hat H = \omega\hat a^\dagger\hat a,\quad \hat x = \hat a,\quad \hat p = i\hat a^\dagger}$$ with cannonical (anti-)comm. rel.

$$\{\hat x,\hat p\} = i \iff \{\hat a,\hat a^\dagger\} = \{\hat a^\dagger,\hat a\} = 1,\quad \{\hat a,\hat a\} = 0 = \{\hat a^\dagger,\hat a^\dagger\}$$

$$\Rightarrow [\hat H,\hat a] \overset{p.68}{=} \omega(-\{\hat a^\dagger,\hat a\}\hat a) = -\omega\hat a,\quad [\hat H,\hat a^\dagger] = \omega\hat a^\dagger\{\hat a,\hat a^\dagger\} = +\omega\hat a^\dagger$$

- We only have 2 states:

the vacuum $|0\rangle$ with $\hat a|0\rangle = 0$

the state $\hat a^\dagger|0\rangle \equiv |1\rangle$

$$\Rightarrow \begin{aligned}\hat a^\dagger|1\rangle &= \hat a^\dagger\hat a^\dagger|0\rangle = 0|0\rangle = 0\\ \hat a|1\rangle &= \hat a\hat a^\dagger|0\rangle = (\hat a^\dagger\hat a+1)|0\rangle = |0\rangle\end{aligned}$$

Grassmann variables

- In the path integral formalism we can replace operators by class variables (at the cost of the path integration), but the latter still have to anti-commute as class. objects.

Axioms of calculation for Grassmann variables c, c^*

- $*$ c, c^* are indep (as x, p or $z, z^* \in \mathbb{C}$) and nilpotent $c^2 = 0 = c^{*2}$
- $*$ $\int dc = 0 = \int dc^*$ formal integration ($\neq$ Riemannian sense)
- $*$ $\int dc\, c = 1 = \int dc^* c^*$ $\quad (\Rightarrow \int dc^* \int dc\, c c^* = 1)$
- $*$ $\{c, c^*\} = cc^* + c^*c$ anticommutation, for more variables $c_{j=1\dots}$

  $\{c_i, c_j\} = 0 = \{c_i, c_j^*\} = \{c_i^*, c_j\}$

  the "differentials" dc, dc^* are also anticom. objects

- $*$ the c_i, c_j^* anticommute with a or a^+.

- $\Rightarrow$ a product of an even number of anticommuting objects is commuting, e.g. ca^+ is commuting (but still nilpotent: $(ca^+)^2 = ca^+ca^+ = -a^+cca^+ = 0$)

- $\Rightarrow$ any function of c_i or c_j can be expanded into a finite Taylor series, e.g. $e^{-ca^+} = 1 - ca^+ + \underbrace{\tfrac{1}{2}(ca^+)^2}_{0} + \underbrace{\dots}_{\text{all } 0} = 1 - ca^+$

2 useful states:

(1) $|c\rangle \equiv e^{-ca^+}|0\rangle = (1 - ca^+)|0\rangle = |0\rangle - c|1\rangle$

$\Rightarrow \hat{a}|c\rangle = +c\hat{a}|1\rangle = c|0\rangle = c|c\rangle$, is a left $\hat{a}$ (or $\hat{x}$) eigenstate

(2) $\langle c| \equiv \langle 0| e^{-\hat{a}c^*} = \langle 0|(1 - \hat{a}c^*) = \langle 0| - \langle 1|c^*$

$\Rightarrow \langle c|\hat{a}^+ = +\langle 1|\hat{a}^+c^* = \langle 0|c^* = \langle c|c^*$ is a right $\hat{a}^+$ (or $\hat{p}$) eigenst.

(3) $\Rightarrow \langle c'|c\rangle = \langle 0|(1 - \hat{a}c'^*)(1 - c\hat{a}^+)|0\rangle = 1 + \langle 0|\hat{a}c'^* c\underbrace{\hat{a}^+|0\rangle}_{|1\rangle} = e^{c'^*c}$

(to (2): note that different conventions exist for complex conj $(c_i c_j) = \pm c_j^* c_i^*$ in the lit.)

- <u>insertion of identity</u> : it holds $\sum_{s=0,1} |s\rangle\langle s| = \mathbb{1}$

alternatively $\int dc^* \int dc\, e^{-c^*c} |c\rangle\langle c| = \int dc^* \int dc\, \underbrace{(1-c^*c)(1-c\hat{a}^\dagger)|0\rangle\langle 0|(1-\hat{a}c^*)}_{\text{collect } \sim c^*c}$

$= \int dc^* \int dc \left(-c^*c|0\rangle\langle 0| + c\hat{a}^\dagger|0\rangle\langle 0|\hat{a}c^*\right)$

$= +\,|0\rangle\langle 0| + |1\rangle\langle 0|$ also serves as $\mathbb{1}$

- <u>complete trace</u> :

for a bosonic operator $\hat{A}$ we have $\mathrm{Tr}[\hat{A}] = \langle 0|\hat{A}|0\rangle + \langle 1|\hat{A}|1\rangle$

alternatively $\int dc^* \int dc\, e^{-c^*c} \langle -c|\hat{A}|c\rangle = \int dc^* \int dc\,(1-c^*c)\langle 0|(1+\hat{a}c^*)\hat{A}(1-c\hat{a}^\dagger)|0\rangle$

$= \int dc^* \int dc\,(-c^*c)\langle 0|\hat{A}|0\rangle - \int dc^* \int dc\, \langle 0|\hat{a}c^*\hat{A}c\hat{a}^\dagger|1\rangle$

$= \langle 0|\hat{A}|0\rangle + \langle 1|\hat{A}|1\rangle$ (note $\mathrm{Tr}[\hat{A}_f]=0$ for $\hat{A}_f$ fermionic)

- we can now proceed as in the bosonic case for $G_p^{(n)}(\tau_1 \ldots \tau_n) =$ with $\beta \geq \tau_1 \geq \tau_2 \geq \ldots \geq \tau_n$

(*) $= \mathrm{Tr}\left[e^{-\beta\hat{H}} T\{\hat{x}_H(\tau_1) \ldots \hat{x}_H(\tau_m)\hat{p}(\tau_{m+1}) \ldots \hat{p}(\tau_n)\}\right]$; $\hat{H} = \omega\hat{a}^\dagger\hat{a}$ bosonic, n even to get $\neq 0$ $\Rightarrow \hat{A}$ is bosonic,

$= \int dc_1^* \int dc_1\, e^{-c_1^*c_1} \langle -c_1| e^{-(\beta-\tau_1)\hat{H}}\hat{x}\, e^{-(\tau_1-\tau_2)\hat{H}} \ldots e^{-(\tau_m-\tau_{m+1})\hat{H}}\hat{p}\, e \ldots |+c_1\rangle$

split $e^{-(\beta-\tau_1)\hat{H}} = \left(e^{-\varepsilon\hat{H}}\right)^{\frac{\beta-\tau_1}{\varepsilon}} = e^{-\varepsilon\hat{H}} \ldots e^{-\varepsilon\hat{H}}$ etc, $\varepsilon = \frac{\beta}{N}$

and insert identities

$e^{-\varepsilon\hat{H}}|c_1\rangle$

$\int dc_N^* \int dc_N\, e^{-c_N^*c_N} \ldots \int dc_2^* \int dc_2\, e^{-c_2^*c_2}$

- <u>s.th. $\hat{x}$ is on the lhs</u> : $e^{-\varepsilon\hat{H}}\hat{x}\, e^{-\varepsilon\hat{H}} = \int dc_j^* dc_j\, e^{-c_j^*c_j}\, e^{-\varepsilon\hat{H}}\hat{x}|c_j\rangle\langle c_j|e^{-\varepsilon\hat{H}}$

to use $\hat{x}|c_j\rangle = c_j|c_j\rangle$

- <u>s.th. $\hat{p}$ is on the rhs</u>: $e^{-\varepsilon\hat{H}}\hat{p}\, e^{-\varepsilon\hat{H}} = \int dc_j^* \int dc_j\, e^{-c_j^*c_j}\, e^{-\varepsilon\hat{H}}|c_j\rangle\langle c_j|\hat{p}\, e^{-\varepsilon\hat{H}}$

to use $\langle c_j|\hat{p} = \langle c_j|ic_j^*$

we have $\hat{H} = \omega \hat{a}^\dagger \hat{a} = -i\omega \hat{p}\hat{x}$ is bosonic :

- $\langle c_{j+1} | e^{-\varepsilon \hat{H}(\hat{p},\hat{x})} | c_j \rangle = \langle c_{j+1} | e^{-\varepsilon H(i c_{j+1}^*, c_j)} | c_j \rangle$ (no approx, $O(\varepsilon^2)$ is zero
- in case of $\hat{x}$ or $\hat{p}$ present we still have

$$\langle c_{j+1} | \hat{p}\, e^{-\varepsilon \hat{H}} | c_j \rangle = \langle c_{j+1} | i c_{j+1}^* \, e^{-\varepsilon H(i c_{j+1}, c_j)} | c_j \rangle \quad \text{as here } O(\varepsilon) \text{ is } 0$$

$$\langle c_{j+1} | e^{-\varepsilon \hat{H}} \hat{x} | c_j \rangle = \langle c_{j+1} | e^{-\varepsilon H(i c_{j+1}, c_j)} c_j | c_j \rangle \quad \text{—— " ——}$$

- $\Rightarrow$ per insertion of $\mathbb{1}$ we have — either $\hat{p}$ or $\hat{x}$ or none is present

$$e^{-c_{j+1}^* c_{j+1}} | c_{j+1} \rangle \langle c_{j+1} | (\hat{p})\, e^{-\varepsilon \hat{H}(\hat{p},\hat{x})} (\hat{x}) | c_j \rangle$$

$|c\rangle$ bosonic

$$= e^{-c_{j+1}^* c_{j+1}} | c_{j+1} \rangle \langle c_{j+1} | c_j \rangle\, e^{-\varepsilon \hat{H}(i c_{j+1}^*, c_j)} \quad (i c_{j+1}^*) (c_j) (-)^{\delta_p + \delta_x}$$

$$= | c_{j+1} \rangle \underbrace{e^{-c_{j+1}^* c_{j+1} + c_{j+1}^* c_j - \varepsilon \hat{H}(i c_{j+1}^*, c_j)}}_{\exp\left[-\varepsilon \left\{ c_{j+1}^* \frac{(c_{j+1} - c_j)}{\varepsilon} + \hat{H}(i c_{j+1}^*, c_j) \right\}\right]} \quad (i c_{j+1}^*)(c_j) (-)^{\delta_p + \delta_x}$$

- starting from the right in ⊛ we can proceed until we reach $\langle -c_1 |$:

$$e^{-c_1^* c_1} \langle -c_1 | e^{-\varepsilon \hat{H}} | c_N \rangle = e^{-c_1^* c_1 - c_1^* c_N - \varepsilon \hat{H}(-i c_1^*, c_N)}$$

$\Rightarrow$ use boundary cond $\underline{c_{N+1} = -c_1}$

$$\Rightarrow ⊛ = \lim_{N\to\infty} \frac{\int dc_1^* \int dc_1 \ldots \int dc_N^* \int dc_N \; c(\tau_1) \ldots c(\tau_m)\, i c(\tau_{m+1}) \ldots i c(\tau_n)\, e^{-S_\varepsilon}}{\int dc_1^* \ldots \int dc_N \; e^{-S_\varepsilon}}$$

$$\text{with } S_\varepsilon = \varepsilon \sum_{j=1}^{N} c_{j+1}^* \frac{c_{j+1} - c_j}{\varepsilon} + H(i c_{j+1}^*, c_j)$$

$$\Rightarrow G_\beta^{(n)}(\tau_1 \ldots \tau_n) = \frac{\int \mathcal{D}c^*(\tau) \int \mathcal{D}c(\tau) \; c(\tau_1) \ldots c(\tau_m) \ldots i c(\tau_n) \exp\left[-\int_0^\beta d\tau \left(c^*(\tau) \frac{dc(\tau)}{d\tau} + H(i c^*(\tau), c(\tau)) \right)\right]}{\int \mathcal{D}c^*(\tau) \int \mathcal{D}c(\tau) \quad \text{—— " ——}}$$

$$\text{with b.c. } \quad c(\beta) = -c(0), \qquad c^*(\beta) = -c^*(0)$$