4.3 Euclidean Dirac Fields

- In 1 dimension we have derived the following path integral

$$\mathrm{Tr}\left[e^{-\beta H(\hat{p},\hat{x})} T\{\hat{X}_H(\tau_1) \ldots \hat{P}_H(\tau_n)\}\right] = \int \mathcal{D}c^* \mathcal{D}c \, c(\tau_1) \ldots ic^*(\tau_n)\, e^{-\int_0^\beta d\tau \left(c^*(\tau)\frac{dc}{d\tau} + H(ic^*, c(\tau))\right)}$$

where $\hat{x} = \hat{a}$, $\hat{p} = i\hat{a}^+$, $\{\hat{x}, \hat{p}\} = i$ operators $\to \{c, c^*\} = 0 = \{c, c\} = \{c^*, c^*\}$ Grassmann

- we can generalise this to QFT for $d=4$ Dirac fields:

$$t \to x^\mu, \quad \hat{x}(t) \to \hat{\Psi}(x^\mu), \quad \hat{p}(t) \to \hat{\Pi}_\Psi = i\hat{\Psi}^+$$

with Hamiltonian $\hat{H} = \int d^3\vec{x}\, \hat{\bar{\Psi}}\left[-i\gamma^j\partial_j + m\right]\hat{\Psi}$; $\hat{\bar{\Psi}} = \hat{\Psi}^+\gamma^0$

– in going to the path integral we replace operators by Grassmann vars.

$\hat{\Psi} \to \Psi = c$, $\hat{\Psi}^+ \to \Psi^+ = c^*$, as a convention we integrate over $\bar{\Psi} = \Psi^+\gamma^0$ instead of $\Psi^+ (= c^*)$

$$\Rightarrow \boxed{\begin{aligned}&\mathrm{Tr}\left[e^{-\beta\hat{H}} T\{\hat{\Psi}_H(x_1) \ldots \hat{\bar{\Psi}}_H(x_n)\}\right] \\ &= \int \mathcal{D}\bar{\Psi} \int \mathcal{D}\Psi\, \Psi(x_1) \ldots \bar{\Psi}(x_n) \exp\left[-\int_0^\beta d\tau \int d^3\vec{x} \left\{\bar{\Psi}\gamma^0\partial_\tau\Psi + \bar{\Psi}\left[-i\gamma^j\partial_j + m\right]\Psi\right\}\right]\end{aligned}}$$

with boundary conditions $\Psi(\beta, \vec{x}) = -\Psi(0, \vec{x})$; $\bar{\Psi}(\beta, \vec{x}) = -\bar{\Psi}(0, \vec{x})$

Euclidean action $\quad e^{-S_E} = e^{-\int_0^\beta d\tau \int d^3\vec{x}\, \mathcal{L}_E}$

- define Euclidean Dirac-γ matrices: $\gamma_0^E \equiv \gamma^0$; $\gamma_j^E \equiv -i\gamma^j$

$\Rightarrow \underline{\{\gamma_\mu^E, \gamma_\nu^E\} = 2\delta_{\mu\nu}}$, $\underline{(\gamma_\mu^E)^+ = \gamma_\mu^E}$ is Hermitian

$\underline{\mathcal{L}_E \equiv \bar{\Psi}\left[\gamma_\mu^E\partial_\mu^E + m\right]\Psi}$, $\underline{\Rightarrow \not{\partial}^E \equiv \gamma_\mu^E\partial_\mu^E = -\not{\partial}^{E+}}$ is anti-Hermitian

[Minkowski action for comparison

Wick rotate $\tau = it$, $\Rightarrow \partial_\tau = -i\partial_t$, $d\tau = i\,dt$

$\Rightarrow e^{-S_E} \to e^{i\int dt \int d^3\vec{x}\, \mathcal{L}_M}$, with $\mathcal{L}_M = \bar{\Psi}\left[i\gamma^\mu\partial_\mu - m\right]\Psi$

Wick's Theorem for Fermions

- after perturbatively expanding an interaction part we have to compute objects $\langle \psi(x_1)\ldots\bar{\psi}(x_n)\rangle_0$, where the average is the path integral with the free Lagrangian $\mathcal{L}_0$

$\Rightarrow$ we need to reduce these to free propagators (contractions), using an analogue Wick theorem for the ψ's

discrete derivation

- consider $I_{k\ldots,l;m\ldots,n} \equiv \int \prod_j dc_j^* dc_j\, c_k\ldots c_l\, c_m^*\ldots c_n^* \exp\{-c_p^* A_{pq} c_q\}$

with $A_{pq}\in\mathbb{C}$ the discrete (bosonic) action. Define Grassmann derivatives as

$$\left\{\frac{d}{dc_i}, c_j\right\} = \frac{d}{dc_i}c_j + c_j\frac{d}{dc_i} = \delta_{ij} = \left\{\frac{d}{dc_i^*}, c_j^*\right\}, \quad \left\{\frac{d}{dc_i},\frac{d}{dc_j}\right\} = 0 = \left\{\frac{d}{dc_j^*},\frac{d}{dc_i^*}\right\}$$

- introduce Grassmann valued sources j, j^*:

$$Z[j,j^*] \equiv \int \prod_i dc_i^* dc_i \exp\{-c_p A_{pq} c_q + c_p^* j_p + j_q^* c_q\}$$

$$\Rightarrow I_{k\ldots,l;m\ldots,n} = \left(\frac{d}{dj_k^*}\right)\ldots\left(\frac{d}{dj_l^*}\right)\left(-\frac{d}{dj_m}\right)\ldots\left(-\frac{d}{dj_n}\right) Z[j,j^*]\Big|_{j=0=j^*}$$

Substitution:

$c_k \to c_k + A^{-1}_{kl} j_l$ where we assume that A^{-1} exists

$c_k^* \to c_k^* + j_l^* A^{-1}_{lk}$ leads to

$$-c^*\cdot A\cdot c + c^* j + j^* c \to -(c^*+j^*A^{-1})A(c+A^{-1}j) + (c^*+j^*A^{-1})j + j^*(c+A^{-1}j)$$
$$= -c^*\cdot A\cdot c + j^*\cdot A^{-1}\cdot j$$

$$\Rightarrow \quad Z[j,j^*] = Z[0,0]\exp\{j_k^* A^{-1}_{kl} j_l\}$$

so diff on the lhs generates vev's, on the rhs it generates powers of A^{-1}

* In analogy to the bosonic case it holds:

- $\langle 1 \rangle_0 = 1$
- for any odd number of c_k and c_l^* we get $\langle c_1^{(*)} \cdots c_{2l+1}^{(*)} \rangle_0 = 0$
- $\langle c_k c_l \rangle_0 = 0 = \langle c_k^* c_l^* \rangle_0$

$$\langle c_k c_m^* \rangle_0 = -\langle c_m^* c_k \rangle = \frac{I_{k,m}}{I} = \frac{d}{dj_k^*}\left(-\frac{d}{dj_m}\right) e^{j_p^* \Lambda^{-1}_{pq} j_q}\Big|_{j=j^*=0} = \Lambda^{-1}_{km}$$

- Ex. 9.1.

$$\langle c_k c_l c_m^* c_n^* \rangle = \underbrace{c_k c_l c_m^* c_n^*}_{} - c_k c_l c_m^* c_n \quad ; \quad c_l c_m^* = \langle c_l c_m^* \rangle_0$$

$$= \langle c_k c_n^* \rangle_0 \langle c_l c_m^* \rangle_0 - \langle c_k c_m^* \rangle_0 \langle c_l c_n^* \rangle_0$$

etc.

$\Rightarrow$ We get the old <u>Wick theorem</u> where we sum over all possible contractions of c's with c^*'s <u>times</u> the sign of perm:

$$\boxed{\langle c_1 \cdots c_n c_1^* \cdots c_n^* \rangle = \sum_{\text{all contr.}} (-)^{\text{perm}} \langle c_{j_1} c_{j_2}^* \rangle_0 \cdots \langle c_{j_n} c_{j_n}^* \rangle_0}$$

- Ex 8.3 generalised to $n \times n$ matrices gives

$$Z[0,0] = \det A$$

<u>Schwinger propagator for the Dirac field</u>

- we need to determine the central building block $\langle c_k c_l^* \rangle_0 \to \langle \psi_\alpha(x) \bar{\psi}_\beta(y) \rangle_0$

$\to$ in Fourier space in finite volume V we have

$$\psi(x) = \frac{1}{V} \sum_P e^{iP\cdot x} \tilde{\psi}(P) \, ; \, \bar{\psi}(x) = \frac{1}{V} \sum_Q e^{-iQ\cdot x} \tilde{\bar{\psi}}(Q)$$

$$\Rightarrow S_E = \int d^4x \, \mathcal{L}_E = \frac{1}{V^2} \sum_{P,Q} \tilde{\bar{\psi}}(Q) \underbrace{V \delta^{(4)}_{P,Q}}_{} \underbrace{[i \gamma_\mu^E P_\mu + m]}_{} \tilde{\psi}(P)$$

$$= \frac{1}{V} \sum_P \tilde{\bar{\psi}}(P) [i \gamma_\mu^E P_\mu + m] \tilde{\psi}(P)$$

$$\Rightarrow \quad \langle \tilde{\Psi}_\alpha(P)\tilde{\bar{\Psi}}_\beta(Q)\rangle = V\,\delta^{(4)}_{P,Q}\,[i\gamma^E_\mu P_\mu + m\,\mathbb{1}_4]^{-1}_{\alpha\beta}$$

- we have $[i\gamma_\mu P_\mu + m\mathbb{1}]^{-1} = \dfrac{-i\gamma^E_\mu P_\mu + m\mathbb{1}}{P^2 + m^2}$, $\gamma^E_\mu P_\mu = \sum_{\mu=1}^{4} \underset{4\times4}{\gamma^E_\mu}\,\underset{\text{scalar}}{P_\mu}$

as $[-i\gamma^E_\nu P_\nu + m][i\gamma^E_\mu P_\mu + m] = \underbrace{\gamma^E_\nu\gamma^E_\mu}_{\frac{1}{2}\{\gamma^E_\nu,\gamma^E_\mu\} = \delta_{\mu\nu}} P_\nu P_\mu + m^2 = P^2 + m^2$

In the $V\to\infty$ limit we have

$$\boxed{\begin{aligned}\langle \tilde{\Psi}_\alpha(P)\tilde{\bar{\Psi}}_\beta(Q)\rangle &= (2\pi)^4\delta^{(4)}(P-Q)\,\frac{[-i\gamma^E_\mu P_\mu + m\mathbb{1}]_{\alpha\beta}}{P^2+m^2}\\ \langle \Psi_\alpha(x)\bar{\Psi}_\beta(y)\rangle &= \int\frac{d^4P}{(2\pi)^4}\,e^{iP\cdot(x-y)}\,\frac{[-i\gamma^E_\mu P_\mu + m\mathbb{1}]_{\alpha\beta}}{P^2+m^2}\end{aligned}}$$

- One can check that this agrees with the direct calculation of $\langle 0|T\{\hat{\Psi}_\alpha(x)\hat{\bar{\Psi}}_\beta(y)\}|0\rangle$ (→ ex.)

- Properties of γ-matrices (here $\gamma^E_\mu = \gamma_\mu$) $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}\mathbb{1}$

- $(\gamma_\mu)^2 = \mathbb{1}$ $\qquad \gamma^E_0 = \begin{pmatrix}0 & \mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}$, $\gamma^E_j = \begin{pmatrix}0 & -i\sigma^j\\ i\sigma_j & 0\end{pmatrix}$

$O(\gamma^0)$ $\Rightarrow \underline{\mathrm{Tr}\,\mathbb{1} = d}$

$O(\gamma)$ $\gamma_\mu\gamma_\nu = -\gamma_\mu\gamma_\nu$ for $\mu\neq\nu \Rightarrow \mathrm{Tr}\,\gamma_\nu = \mathrm{Tr}\,\gamma_\mu^2\gamma_\nu = -\mathrm{Tr}\,\gamma_\mu\gamma_\nu\gamma_\mu \underset{\text{cyclic}}{=} \underline{-\mathrm{Tr}\,\gamma_\nu = 0}$

$O(\gamma^2)$ $\mathrm{Tr}[\gamma_\mu\gamma_\nu] = \mathrm{Tr}\,\mathbb{1}\,\delta_{\mu\nu} = d\,\delta_{\mu\nu}$

$O(\gamma^3)$ $\mathrm{Tr}[\gamma_\alpha\gamma_\beta\gamma_\delta] = \begin{cases}\alpha=\beta & = \mathrm{Tr}[\gamma_\delta] = 0\\ \alpha\neq\beta\neq\delta\neq\alpha & = \mathrm{Tr}[\underset{\mu\neq\text{all other}}{\gamma_\mu^2}\gamma_\alpha\gamma_\beta\gamma_\delta] = -\mathrm{Tr}[\gamma_\mu\gamma_\alpha\gamma_\beta\gamma_\delta\gamma_\mu] \overset{\text{cyclic}}{=} -\mathrm{Tr}[\gamma_\alpha\gamma_\beta\gamma_\delta] \\ & \underline{= 0}\end{cases}$

$O(\gamma^4)$ $\mathrm{Tr}[\gamma_\mu\gamma_\nu\gamma_\beta\gamma_\delta] = \ldots$
$= d(\delta_{\mu\nu}\delta_{\beta\delta} - \delta_{\mu\beta}\delta_{\nu\delta} + \delta_{\mu\delta}\delta_{\nu\beta})$

Define $\gamma^5 \equiv \gamma_0\gamma_1\gamma_2\gamma_3$, $\gamma_5^2 = 1$, $\gamma_5^\dagger = \gamma_5$, more ex 9.2