5. Gauge fields

- After treating scalar fields (spin 0) and Dirac Fermions (spin $\frac{1}{2}$) we will now consider <u>vector fields</u>, e.g. $A^\mu(x)$ vector potential

<u>Motivation:</u> – in particle physics interactions are mediated by vector fields (photons, Z- & W-bosons, gluons)

– if we want to extend <u>global symmetries</u>,

"gauging a symmetry" e.g. $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} \to U\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix}$ ($U \in SO(2)$ with U x-indep (see ex. 1.4 for current)

to <u>local symmetries</u> $U \to U(x)$, space-time dep.

we need to introduce vector fields to achieve a (gauge) invariant Lagrangian!

→ gauge (vector) fields couple "matter" fields $\phi(x)$, $\psi(x)$!

Note that we could also consider pure gauge theories (ED, YM), however the concept of local symmetries of invariances of that theories remain.

<u>Lorentz invariance $\Lambda^\mu{}_\nu$</u>

We have $x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu$, $A^\mu(x) \to A'^\mu(x') = \Lambda^\mu{}_\nu A^\nu(x)$
for vectors

$$\Rightarrow \boxed{\mathcal{L}_M = \underbrace{\frac{1}{2}\partial^\mu A^\nu \partial_\mu A^\nu}_{\text{Maxwell}} \underbrace{- \frac{m^2}{2} A^\mu A_\mu}_{\text{mass}}} \quad \text{(}\to \text{Proca e.o.m) is invariant } \left((\partial^\mu A_\mu)^2 \text{ too} \right)$$

- while A^μ (4 comp.) is handy to formulate ED in an invariant way the photon has only 2 d.o.f. helicities $\pm$ ⇒ 2 d.o.f redundant

→ in classical <u>and</u> quant. theories we have to reduce to the physical # d.o.f ("fixing a gauge")

concept of enlargement of # d.o.f still usefull !

5.1. Building an invariant action for local trafos: $U(1)$

- Example: 2 scalar fields $\phi_{1,2}$ or $\Phi = \phi_1 + i\phi_2$

$$\mathcal{L}_M = (\partial_\mu \phi_j)(\partial^\mu \phi_j) - m^2 \phi_j \phi_j = \partial_\mu \Phi^* \partial^\mu \Phi - m^2 \Phi^* \Phi$$

global trafo: $\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} \cos\omega & +\sin\omega \\ -\sin\omega & \cos\omega \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \iff \Phi' = e^{i\omega}\Phi$, $\Phi'^* = e^{-i\omega}\Phi^*$ (we did $\omega \ll 1$ infinitesimal in ex 1.2)

$SO(2)$ or $U(1)$ trafo, $\mathcal{L}_M' = \mathcal{L}_M$ invariant

→ local trafo: $\omega \to \omega(x)$ = gauge trafo

$$\boxed{\Phi' = e^{+i\omega(x)}\Phi, \quad \Phi'^* = e^{-i\omega(x)}\Phi^*(x)}$$

⇒ mass term $\Phi^*\Phi$ invariant due to $e^{-i\omega(x)} e^{i\omega(x)} = 1$

BUT: the kinetic term is not invariant

infinitesimal $\omega(x) \ll 1$:

$$\partial_\mu \Phi^* \partial^\mu \Phi \to \partial_\mu (e^{-i\omega(x)}\Phi^*) \partial^\mu (e^{+i\omega(x)}\Phi) = \partial_\mu \Phi^* \partial^\mu \Phi + (\partial_\mu \omega(x)) J^\mu + \mathcal{O}(\omega^2)$$

with $J^\mu = i\Phi^* \partial^\mu \Phi - i\partial^\mu \Phi^* \Phi = 2(\phi_1 \partial^\mu \phi_2 - \partial^\mu \phi_1 \phi_2)$ conserved current for ω const!

if we add a field coupling to J^μ that counteracts:

$$\mathcal{L}_M + g J^\mu A_\mu(x), \quad \boxed{A_\mu(x) \to A_\mu(x) + g^{-1}\partial_\mu \omega(x)}$$

inhomog. trafo! for $\omega \ll 1$

↑ this is invariant under local trafos! ↑ g coupling

- for A_μ to be a dynamical (not auxiliary) field we add an inv. + kinetic term:

$$\mathcal{L}_M + g J^\mu A_\mu - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \mathcal{L}_{tot}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \xrightarrow[\text{trafo}]{} \partial_\mu A_\nu + g^{-1}\partial_\mu \partial_\nu \omega(x) - \partial_\nu A_\mu - g^{-1}\partial_\nu \partial_\mu \omega(x)$

$= F_{\mu\nu}$ invariant (for $\omega(x)$ smooth)

- note a vector field $\hat{A}_\nu = \partial_\nu \Omega(x)$ is called "pure gauge"

- the mass term for the vector field is not gauge invariant!

- We can write for $\omega(x)$ general

$$\underline{\mathcal{L}_{tot}} = \underbrace{(\partial_\mu \Phi^*(x) + ig A_\mu(x) \Phi^*)}_{(D_\mu \Phi)^*} \underbrace{(\partial^\mu \Phi(x) - ig A_\mu(x) \Phi(x))}_{D_\mu \Phi(x)} - m^2 \Phi^*(x)\Phi(x) \; \underline{- \frac{1}{4} F^{(x)}_{\mu\nu} F^{\mu\nu}_{(x)}}$$

with covariant derivative $\boxed{D_\mu = \partial_\mu - ig A_\mu} \Rightarrow (D_\mu \Phi)' = D_\mu \Phi$

$\rightsquigarrow$ this concept can be generalised to larger local sym. groups and Ψ's.

– <u>Building gauge-invariant actions for $SU(N)$</u>

local trafo $\quad \phi(x) \to \phi'(x) = U(x)\, \phi(x) \qquad U(x) \in SU(N)$

$\qquad \Psi_\alpha(x) \to \Psi'_\alpha(x) = U(x)\, \Psi_\alpha(x)$, $\quad \phi, \Psi$ complex N-vector

$\Rightarrow$ <u>mass terms</u> $\phi^\dagger \phi$, $\bar\Psi \Psi = \Psi^\dagger_\alpha (\gamma_0)_{\alpha\beta} \Psi_\beta$ <u>invariant</u> ($U(x)$ commutes with γ_0)

<u>covariant derivative</u>

summation conv.

$\boxed{D_\mu = \partial_\mu - ig A_\mu(x)}$, $\quad A_\mu(x) = A^a_\mu(x)\, T^a$, $\underline{A^a_\mu(x) \in \mathbb{R}}$ $\quad T^{a = 1, \dots, N^2-1}$ generators of $SU(N)$

it holds: $(T^a)^\dagger = T^a$, $\operatorname{Tr} T^a = 0$, $\operatorname{Tr}[T^a T^b] = \frac{1}{2}\delta^{ab}$, $[T^a, T^b] = i f^{abc}\, T^c$

$\Rightarrow A^\dagger_\mu = A_\mu$ $\qquad$ Structure const., Jacobi Id.

<u>gauge trafo</u>:

$$A_\mu(x) \to \boxed{A'_\mu(x) = U(x) A_\mu(x) U^\dagger(x) + ig^{-1} U(x)\, \partial_\mu U^\dagger(x)} \qquad 0 = \partial_\mu(U U^\dagger) = \partial_\mu U U^\dagger + U \partial_\mu U^\dagger$$

$$\Rightarrow A'^\dagger_\mu(x) = U A_\mu U^\dagger - ig^{-1} \partial_\mu U\, U^\dagger \overset{!}{=} A'_\mu(x)$$

[We have $\operatorname{Tr} A_\mu = 0 \to \operatorname{Tr} A'_\mu = \operatorname{Tr}(U A_\mu U^\dagger + ig^{-1} U \partial_\mu U^\dagger) = \underbrace{\operatorname{Tr}(A_\mu)}_{=0} + ig^{-1} \underbrace{\operatorname{Tr}(U \partial_\mu U^\dagger)}_{=0}$

as $U(x) = e^{i\varepsilon^a(x) T^a} \Rightarrow \partial_\mu U U^\dagger$ is traceless

consistent!]

$$\Rightarrow (D_\mu \phi)' = (\partial_\mu - ig\,\mathcal{U} A_\mu \mathcal{U}^+ + \mathcal{U}\partial_\mu \mathcal{U}^+)\mathcal{U}\phi$$
$$= \mathcal{U}(\partial_\mu - igA_\mu)\phi + \underbrace{(\partial_\mu \mathcal{U}\,\mathcal{U}^+ + \mathcal{U}\partial_\mu \mathcal{U}^+)}_{=0}\mathcal{U}\phi = \mathcal{U} D_\mu \phi$$
$$\Leftrightarrow D_\mu' \mathcal{U} = \mathcal{U} D_\mu \Leftrightarrow \underline{D_\mu' = \mathcal{U} D_\mu \mathcal{U}^+}$$

$$\Rightarrow \boxed{\begin{aligned} \mathcal{J}_\mu(\phi) &= (D^\mu\phi)^+ D_\mu\phi - m\,\phi^+\phi \\ \text{and ditto} \quad \mathcal{J}_\mu(\psi) &= \bar{\psi}_\alpha\left[i\gamma^\mu_{\alpha\beta} D_\mu - \delta_{\alpha\beta}\right]\psi_\beta \end{aligned}}$$ are invariant

Q: kinetic term for the vector fields?

Define $F_{\mu\nu} = F^a_{\mu\nu} T^a \equiv \frac{i}{g}[D_\mu, D_\nu] = \frac{i}{g}[\partial_\mu - igA_\mu, \partial_\nu - igA_\nu]$

$\Rightarrow$ gauge trafo $F'_{\mu\nu} = \frac{i}{g}[D'_\mu, D'_\nu] = \frac{i}{g}[\mathcal{U} D_\mu \mathcal{U}^+, \mathcal{U} D_\nu \mathcal{U}^+] = \mathcal{U} F_{\mu\nu} \mathcal{U}^+$

$\Rightarrow \mathrm{Tr}\, F^{\mu\nu} F_{\mu\nu}$ is gauge and Lorentz invariant!

components:

$$\underline{F^a_{\mu\nu}} = 2\,\mathrm{Tr}[T^a F^b_{\mu\nu} T^b] = \frac{2i}{g}\mathrm{Tr}\left[T^a\left(\underset{0}{[\partial_\mu, \partial_\nu]} - ig[\partial_\mu A_\nu] - ig[A_\mu, \partial_\nu] - g^2[A_\mu A_\nu]\right)\right]$$
$$= \frac{2i}{g}\mathrm{Tr}\left[T^a\left(-ig(\partial_\mu A^b_\nu(x))T^b + ig(\partial_\nu A^b_\mu(x))T^b - g^2 A^b_\mu A^c_\nu \underbrace{[T^b, T^c]}_{i f^{bcd} T^d}\right)\right]$$
$$= \underline{\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{bca} A^b_\mu A^c_\nu}$$

$\Rightarrow \quad \mathcal{J}_\mu(\psi) - \frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu}$ inv. theory: $U(1) = $ QED, $SU(3) = $ QCD

$\mathcal{J}_\mu(\phi) - \frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} - V(\phi^*\phi)$ SU(2): Higgs + weak interact.