5.2. Quantisation of gauge fields : U(1)

- in principle we could proceed with the path integral quant. of $A_\mu(x)$, treating each component $A_{\mu=0,1,2,3}$ as an indep. scalar field:

reminder: for 1-dim QM in Euclid. space time we had

$$\mathrm{Tr}\left[e^{-\beta H}\right] = \int \mathcal{D}x \int \mathcal{D}p \exp\left[-\int_0^\beta d\tau \left(H(x,p) + ip\dot{x}\right)\right] \xrightarrow[\text{p's}]{\text{int. out}} \int \mathcal{D}x \exp\left[-\int_0^\beta d\tau\, L(x,\dot{x})\right]$$

- the second step is only allowed for a theory quadratic in the momenta p, without constraints (→ stick to $H(x,p)$ or Fadeev-Popov trix, later)

Here this is not the case: A_μ has 4 comp., the photon 2 →

⇒ partly fix the gauge = set field comp to 0 and/or constraints

another facet of this problem

in the fully Lorentz- and gauge inv. theory the Green's fuct. (propagator) does not exist:

pure U(1) gauge theory (Maxwell): $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

class. eq. of motion $\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$

$$\Leftrightarrow \boxed{(g_{\mu\nu} \underset{\partial_\beta \partial^\beta}{\Box} - \partial_\mu \partial_\nu) A^\mu = 0} \quad (*)$$

after integration by parts (disregarding surface terms): $\boxed{\mathcal{L} = +\frac{1}{2} A^\mu \left[g_{\mu\nu} \Box - \partial_\mu \partial_\nu\right] A^\nu}$

⇒ Green's fuct $D_{\mu\nu}$: $(g_{\mu\nu}\Box - \partial_\mu\partial_\nu) D^{\nu\lambda}(x) = \delta_\mu^\lambda \delta^{(4)}(x)$

BUT: the diff. op. has a nonvanishing kernel $A^\nu = \partial^\nu \Lambda(x)$ (pure gauge)

as $(g_{\mu\nu}\Box - \partial_\mu\partial_\nu)\partial^\nu \Lambda(x) = (\Box\partial_\mu - \partial_\mu\Box)\Lambda(x) = 0 \quad \forall$ smooth $\Lambda(x)$

to define $D^{\nu\lambda}$ ($\hat{=} \sim \langle 0|T\{A^\nu A^\lambda\}|0\rangle$) we need to fix the gauge

Gauge fixing

- by add a gauge fixing $\mathcal{L}_{GF}$ to $\mathcal{L}$ we can modify the quadratic term to obtain an invertible diff. op

(the resulting theory should of course still give Maxwells eq. for $\vec{E}$ and $\vec{B}$-field! at the end)

there are several possibilities:

- Lorentz gauge $\partial_\mu A^\mu = 0$ $\Rightarrow$ in e.o.m (*) gives $g_{\mu\nu} \Box A^\mu = 0$

  this yields an invertible diff. op

  However is reduces only 4 -> 3 d.o.f , a further reduction is called

- radiation or Coulomb gauge

  $A_0 \equiv 0$, $\partial_j A_j = 0$ this has 2 d.o.f as the photon

- axial gauge

  $A_3 \equiv 0$, express A_0 and it's derivatives through $A_{2,3}$ (constraints)

- parameter dependent gauge $\mathcal{L}_{GF} = -\frac{1}{2\alpha}(\partial_\mu A^\mu)^2$, α finite

$\Rightarrow \mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 \overset{\text{p.int}}{=} \frac{1}{2} A^\mu \left[g_{\mu\nu}\Box - \left(1 - \frac{1}{\alpha}\right)\partial_\mu \partial_\nu \right] A^\nu$

so $\alpha = 1$ is Lorentz gauge (and still = same e.o.m) * only $\alpha = 1$, $\alpha \to 0$ give Maxwell's eq.

$\Rightarrow$ in momentum space we have for the diff. op:

$O_{\mu\nu} = -g_{\mu\nu} k^\rho k_\rho + \left(1 - \frac{1}{\alpha}\right) k_\mu k_\nu$. This yields a propagator

$D_{\mu\nu} = -\frac{1}{k^2}\left[g_{\mu\nu} + (\alpha - 1)\frac{k_\mu k_\nu}{k^2} \right]$ with $D_{\mu\nu}(k)\, O^{\nu\rho} = \delta_\mu^\rho$ (ex 11

which yields the Green's function after Fourier trafo.

Here we can take $\lim \alpha \to 0$ which is called Landau gauge.

Canonical quantisation in Lorentz gauge

for radiation gauge see [R] ch 4.4

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{2} (\partial_\rho A^\rho)^2$$

we try to follow the canonical quantisation for scalar fields (applied to A_μ)

$$\boxed{\begin{aligned} &[A_\mu(x^0, \vec{x}), \pi_\nu(x^0, \vec{y})] = i g_{\mu\nu} \delta^{(3)}(\vec{x} - \vec{y}) \\ &[A_\mu(x^0, \vec{x}), A_\nu(x^0, \vec{y})] = 0 = [\pi_\mu(x^0, \vec{x}), \pi_\nu(x^0, \vec{y})] \end{aligned}}$$

with conj momenta $\quad \pi^\mu \equiv \dfrac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = \underbrace{-\partial^0 A^\mu + \partial^\mu A^0}_{-F^{0\mu}} - g^{0\mu} \partial_\rho A^\rho$

- in particular we have $\pi^0 = \dfrac{\partial \mathcal{L}}{\partial(\partial_0 A_0)} = -\partial_\rho A^\rho$, $\quad \pi^i = \partial^i A^0 - \partial^0 A^i$

before addin $\mathcal{L}_{GF}$ this was vanishing, causing a problem (which we could fix in Coulomb gauge having $A^0 = 0 = \pi^0$)

$\rightarrow$ we cannot impose Lorentz gauge as a condition on the op $\partial_\rho A^\rho \equiv 0$, we'll only impose it on expect. values of (phys.) states $\langle \psi | \partial_\rho A^\rho | \psi \rangle = 0$

quantise $\quad A_\mu(x) = \displaystyle\int \frac{d^3 p}{\sqrt{(2\pi)^3 2E_p}} \left\{ \sum_{\lambda=0}^{3} \varepsilon_\mu^{(\lambda)}(\vec{p}) \left[\hat{a}_{\vec{p}}^{(\lambda)} e^{-ip\cdot x} + \hat{a}_{\vec{p}}^{(\lambda)\dagger} e^{ip\cdot x} \right] \right.$

with 4 polarisation vectors $\varepsilon_\mu^{(\lambda)}$, $\lambda = 0,1,2,3$, solutions of $\Box A_\mu = 0$ in plane waves

we choose a Lorentz inv normalisation

$$\varepsilon_\mu^{(\lambda)} \varepsilon^{(\rho)\mu} = g^{\lambda\rho}$$

- not all polarisations will be physical: e.g. for a photon moving $\parallel e_3$ $\; k^\mu = \begin{pmatrix} k \\ 0 \\ 0 \\ k \end{pmatrix}$

$\varepsilon^{(\lambda)} = e_\lambda$, the transverse pol. $k \cdot \varepsilon^{(\lambda=1,2)} = 0$ are physical

$\varepsilon^{(0)}$ is called scalar (or timelike), $\varepsilon^{(3)}$ longitudinal

- from the equal time commutators we get by differentiation wrt ∂_j

$$\left[\frac{\partial}{\partial x_j} A_\mu(x^0,\vec{x}), A_\nu(x^0,\vec{y})\right] = 0 \qquad j = 1,2,3$$

$$\Rightarrow \left[A_\mu(x^0,\vec{x}), \pi_\nu(x^0,\vec{y})\right] = \left[A_\mu(x^0,\vec{x}), -\partial_0 A_\nu(x^0,\vec{y})\right] = i g_{\mu\nu}\, \delta^{(3)}(\vec{x}-\vec{y})$$

inserting the expression for $A_\mu(x)$ in terms of $\hat{a}, \hat{a}^+$'s we get

$$\underline{\left[\hat{a}^{(\lambda)}_{\vec{p}}, \hat{a}^{(\varrho)+}_{\vec{q}}\right] = -g^{\lambda\varrho}\,\delta^{(3)}(\vec{p}-\vec{q})} \quad \text{and } 0 = \left[\hat{a}^{(\lambda)}_{\vec{p}}, \hat{a}^{(\varrho)}_{\vec{q}}\right] = \left[\hat{a}^{+(\lambda)}_{\vec{p}}, \hat{a}^{+(\varrho)}_{\vec{q}}\right]$$

because of $g^{\mu\nu} = \begin{pmatrix} 1 & & 0 \\ & -1 & \\ 0 & & -1_1 \end{pmatrix}$ these are the same comm. rel. as for scalar fields for $\lambda, \varrho = 1,2,3$

BUT: the scalar photons created negative norm states.

define $|1_0\rangle \equiv \int \frac{d^3\vec{p}}{\sqrt{(2\pi)^3 2E_{\vec{p}}}} f(\vec{p})\, \hat{a}^{(0)+}_{\vec{p}} |0\rangle$, with $|0\rangle$ the vacuum

$$\Rightarrow \langle 1_0|1_0\rangle = \int_{\vec{p}}\int_{\vec{q}} f^*(\vec{q}) f(\vec{p}) \langle 0| \underbrace{\hat{a}^{(0)}_{\vec{q}}\, \hat{a}^{(0)+}_{\vec{p}}}_{= -\hat{a}^{(0)+}_{\vec{p}}\hat{a}^{(0)}_{\vec{q}} - \delta^{(3)}(\vec{p}-\vec{q})} |0\rangle = -\int \frac{d^3\vec{p}}{(2\pi)^3 2E_{\vec{p}}} |f(\vec{p})|^2 \underbrace{\langle 0|0\rangle}_{1} < 0$$

- likewise we obtain for the

<u>Hamiltonian</u> $\hat{H} = \int d^3\vec{p}\, E_{\vec{p}} \left[\sum_{\lambda=1}^{3} \hat{a}^{(\lambda)+}_{\vec{p}} \hat{a}^{(\lambda)}_{\vec{p}} - \hat{a}^{(0)+}_{\vec{p}} \hat{a}^{(0)}_{\vec{p}}\right]$

it can have negative expectation values :

$$\langle 1_0|\hat{H}|1_0\rangle = \langle 1_0| \int d^3\vec{p}\, E_{\vec{p}} (-) \hat{a}^{(0)+}_{\vec{p}} \hat{a}^{(0)}_{\vec{p}} \int_{\vec{q}} f(\vec{q})\, \hat{a}^{(0)+}_{\vec{q}} |0\rangle$$

$$= \langle 1_0| \int d^3\vec{p}\, E_{\vec{p}} f(\vec{p}) (-) \hat{a}^{(0)+}_{\vec{p}} |0\rangle$$

$$= -\int \frac{d^3\vec{p}}{(2\pi)^3 2} |f(\vec{p})|^2 \langle 0|0\rangle < 0$$

Way out: Lorentz gauge condition

- we cannot require $\partial_\mu \hat{A}^\mu(x)|\Psi\rangle = 0$

on physical states as this is not even satisfied by $|\Psi\rangle = |0\rangle$, as $\partial_\mu \hat{A}^\mu$ contains $\hat{a}_{\vec{p}}^{(\lambda)\dagger}$'s

$\Rightarrow$ require phys states to satisfy $\boxed{\underbrace{\partial_\mu \hat{A}^\mu(x)}_{\text{only } \hat{a}\text{'s}} |\Psi\rangle = \underbrace{\int_{\vec{p}} (-ip_\mu) \sum_{\lambda=0}^{3} \varepsilon^{\mu(\lambda)}_{(\vec{p})} \hat{a}^{(\lambda)}_{\vec{p}} e^{-ip\cdot x}}_{=\partial_\mu \hat{A}^{(+)\mu}} |\Psi\rangle = 0}$, $\partial_\mu \hat{A}^{(-)\mu}$ for $\hat{a}^\dagger$ only

$\Rightarrow$ expectation values vanish

$$\langle\Psi|\partial_\mu \hat{A}^\mu|\Psi\rangle = \langle\Psi|(\underbrace{\partial_\mu \hat{A}^{(+)\mu}}_{\text{gives } 0} + \overbrace{\partial_\mu \hat{A}^{(-)\mu}}^{\text{gives } 0})|\Psi\rangle = 0 \quad \text{on phys. states}$$

- as transversal photons satisfy $p_\mu \varepsilon^{\mu(\lambda=1,2)} = 0$

this translates into

$$\Big(p_\mu \varepsilon^{\mu(0)}_{(\vec{p})} \hat{a}^{(0)}_{\vec{p}} + \underbrace{p_\mu \varepsilon^{\mu(3)}_{(\vec{p})}}_{=-p_\mu \varepsilon^{\mu(0)}_{(\vec{p})}} \hat{a}^{(3)}_{\vec{p}}\Big)|\Psi\rangle = 0$$

$\Rightarrow$ On physical states the effects of scalar and longitudinal photons cancel.

It can be shown that the subset of physical states is a bona fide Fock space of positive norm states only [Gupta - Bleuler formalism]

- In the Faddeev - Popov or BRST approach for non-abelian gauge theories we shall introduce extra fields, so-called ghosts, which cancel the effect of unphysical photons (scalar & long). The ghost will be scalar fields with fermionic statistics (Grassmann)