5.3. Faddeev - Popov method and ghosts

U(1) gauge theory:

- naive path integral without gauge fixing

$$Z = \int \mathcal{D}A_\mu \exp\left[i \int d^4x \, \mathcal{L}_M\right] \quad \text{is ill defined:}$$

denote by A_μ all gauge fields that can be reached from a fixed $\bar{A}_\mu$ by a gauge trafo $\Lambda(x)$: $A_\mu = \tilde{A}_\mu + \partial_\mu \Lambda(x)$

$\Rightarrow \int \mathcal{D}A_\mu = \int \mathcal{D}\tilde{A}_\mu \int \mathcal{D}\Lambda(x)$. Due to $\mathcal{L}_M$ being $\Lambda(x)$-indep. the latter is ill def $\rightarrow$ replace by an integral incl. gauge fixing:

$$Z = \int \mathcal{D}\tilde{A}_\mu \int \mathcal{D}\Lambda(x) \, e^{i \int d^4x \left(\mathcal{L}_M - \frac{1}{2\alpha} G^2\right)} \det\left[\frac{\partial G}{\partial \Lambda}\right]$$

where $G = G(A_\mu)$ is the gauge fixing fcn, e.g. $G = \partial_\mu A^\mu(x)$.
The det is needed to make the integral invariant under $G \to G'(A)$
($\Rightarrow$ Jacobian $\det\left(\frac{\partial G'}{\partial G}\right)$)

- we can use Grassmann variables $\{c^*(x), c(x)\}$ to raise $\det\left[\frac{\partial G}{\partial \Lambda} \equiv M\right]$ into the action

$$\Rightarrow \boxed{Z = \int \mathcal{D}A_\mu \int \mathcal{D}c^* \mathcal{D}c \exp\left[i \int d^4x \left(\mathcal{L}_M - \frac{1}{2\alpha} G^2 - c^{*T} M c\right)\right]}$$

$c(x)$: scalar, anti-comm. fields: "ghosts"*

for the parameter dependent Lorentz gauge we can determine M:

gauge trafo $G(A_\mu) = \partial_\mu A^\mu(x) \rightarrow \partial_\mu\left(A^\mu + \partial^\mu \Lambda(x)\right)$

$$\Rightarrow \frac{\partial G}{\partial \Lambda(x)} = \Box$$

- this concept generalizes to the non-abelian case.

* here: M does not couple to A_μ $\rightsquigarrow$ we can integrate out the c's and absorb $\det \Box$ into the normalization constant of Z

Note: The choice of gauge fixing cond. G does not necessarily specify $\tilde{A}_\mu$ uniquely $\to$ Gribov copies for $SU(N_c)$

Generalisation to the non-abelian case $SU(N)$

$$Z = \int \mathcal{D}A_\mu \int \mathcal{D}\bar{c}^a \int \mathcal{D}c^a \exp\left[i\int d^4x \left(\mathcal{J}_\mu - \frac{1}{2\xi} G^a G^a - \bar{c}^a M_{ab} c^b + \bar{\Psi}(i\gamma^\mu D_\mu - m)\Psi\right)\right]$$

where $\mathcal{J}_\mu = -\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu}$, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{bca} A^b_\mu A^c_\nu$

$$D_\mu = \partial_\mu - ig A_\mu(x) = \partial_\mu - ig A^a_\mu(x) T^a$$

- the particular case $\boxed{G^a = -\partial^\mu A^a_\mu}$ is called covariant gauge (fixing)

for which we will compute $M_{ab} = \frac{\partial G^a}{\partial \theta^b}$

- gauge trafo $A'_\mu = U A_\mu U^\dagger - \frac{i}{g} \partial_\mu U U^\dagger$ $\quad U \in SU(N)$

parametrisation: $U = e^{ig\theta} = \exp[ig T^a \theta^a(x)] \Rightarrow M_{ab} = \frac{\partial G_a}{\partial \theta^b}$

consider infinitesimal trafo $A_\mu \to A'_\mu$:

- $U A_\mu U^\dagger = e^{ig\theta} A_\mu e^{-ig\theta} = A_\mu + ig[\theta, A_\mu] + O(\theta^2) = A^a_\mu T^a + ig[\theta^b T^b, A^c_\mu T^c] + O(\theta^2)$

$$= T^a \left(A^a_\mu - g f^{bca} \theta^b A^c_\mu(x)\right) + O(\theta^2)$$

- $-\frac{i}{g}\partial_\mu U U^\dagger = -\frac{i}{g} ig (\partial_\mu \theta) U U^\dagger = T^a \partial_\mu \theta^a(x)$

$$\Rightarrow A^{a\prime}_\mu = A^a_\mu + \partial_\mu \theta^a(x) - g f^{bca} \theta^b A^c_\mu + O(\theta^2) = A^a_\mu + \frac{\partial A^a_\mu}{\partial \theta^b} \theta^b + O(\theta^2)$$

$$\Rightarrow \frac{\partial G^a}{\partial \theta^b} = \frac{\partial G^a}{\partial A^d_\mu} \frac{\partial A^d_\mu}{\partial \theta^b} = -\partial^\mu\left(\partial_\mu \delta^{ab} - g f^{bca} A^c_\mu\right) = M_{ab}$$

$\Rightarrow$ after integration by parts $\int d^4x\, \bar{c}^a M_{ab} c^b = \int d^4x\, \partial^\mu \bar{c}^a \partial_\mu c^a - \partial_\mu \bar{c}^a g f^{bca} A^c_\mu c^b$

- in general then the ghost couple to A_μ^c ⇒ cannot integrate out like in QED

exception:

Axial gauge: $\boxed{t^\mu A_\mu^a = 0}$ for $t^\mu t_\mu = -1$ fixed spacelike vector

(this holds not only on physical states)

choose $G^a = t^\mu A_\mu^a$ & integrate out c's

⇒ modified propagator $-\frac{1}{4k^2}\left[g^{\mu\nu} + \frac{(t^2+\xi k^2)k^\mu k^\nu}{(k\cdot t)^2} - \frac{k^\mu t^\nu + t^\mu k^\nu}{(k\cdot t)}\right]$

summary what we have done:

- We have inserted the FP identity $\mathbb{1} = \det[M] \int \mathcal{D}\theta \, \delta(G^a[A_\mu, \theta] - g^a(x))$
- we may shift the δ-function by a const., θ-indep field $g^a(x)$
- we may average lhs and rhs by a Gaussian integral

$\int \mathcal{D}g \exp\left[\frac{-i}{2\xi}\int d^4x \, g^a(x) g^a(x)\right]$

$$\Rightarrow Z \to Z = \int \mathcal{D}A_\mu \, e^{i\int d^4x \mathcal{L}_{ee}} \det[M] \int \mathcal{D}\theta \, e^{-\frac{i}{2\xi}\int d^4x \, G^a[A] G^a[A]}$$

gauge inv of group int [details [R] ch 7.2]

decoupling as (∞) const.

The Euclidean action $S_E(A_\mu^a, c^a, \Psi)$ after Fourier trafo:

quadratic part of $\mathcal{L}_E = \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^a + \bar{\Psi}(\gamma_\mu^E D_\mu + m)\Psi + \frac{1}{2\xi} G^a G^a + \bar{c}^a \frac{\partial G^a}{\partial \Theta^b} c^b$

$$\tilde{S}_E = \int \frac{d^4 P}{(2\pi)^4} \int \frac{d Q^4}{(2\pi)^4} \delta^{(4)}(P+Q) \Big\{ \frac{1}{2} i P_\mu \tilde{A}_\nu^a(P) \big(i Q_\mu \tilde{A}_\nu^a(Q) - i Q_\nu \tilde{A}_\mu^a \big) + \frac{1}{2\xi} i P_\mu \tilde{A}_\mu^a(P)\, i Q_\nu \tilde{A}_\nu^a(Q) \Big\}$$

$$+ \delta(P-Q) \Big\{ \tilde{\bar{\Psi}}_{\alpha A}(P) \big[\gamma_\mu^E i Q_\mu + m \big]_{\alpha\beta} \tilde{\Psi}_{\beta A}(Q) - i P_\mu \tilde{\bar{c}}^a(P)\, i Q_\mu \tilde{c}^a(Q) \Big\}$$

recall $\Psi \to \Psi'_A = U_{AB} \Psi_B$ $\quad A, B$ $SU(N)$ indices

$$= \int \frac{d^4P\, d^4Q}{(2\pi)^8} \delta^{(4)}(P+Q) \frac{1}{2} \tilde{A}_\mu^a(P) \tilde{A}_\nu^a(Q) \Big[P^2 \delta_{\mu\nu} - \Big(1 - \frac{1}{\xi}\Big) P_\mu P_\nu \Big]$$

$$+ \delta^{(4)}(P-Q) \Big\{ \tilde{\bar{\Psi}}_{\alpha A}(P) \big[i\gamma_\mu^E P_\mu + m \big]_{\alpha\beta} \tilde{\Psi}_{\beta A}(Q) + \tilde{\bar{c}}^a(P) \tilde{c}^a(Q) P^2 \Big\}$$

Feynman Rules:

- propagators: $\underbrace{\tilde{A}_\mu^a(P) \tilde{A}_\nu^b(Q)} = \delta^{ab} \delta^{(4)}(P+Q) \frac{1}{P^2} \Big(\delta_{\mu\nu} - (1-\xi) \frac{P_\mu P_\nu}{P^2} \Big)$

$\underbrace{\tilde{c}^a(P) \tilde{\bar{c}}^b(Q)} = \delta^{ab} \delta^{(4)}(P-Q) \frac{1}{P^2}$ as for scalar fields p. 57

$\underbrace{\tilde{\Psi}_{\alpha A}(P) \tilde{\bar{\Psi}}_{\beta B}(Q)} = \delta_{AB} \delta^{(4)}(P-Q) \frac{[-i\gamma_\mu^E P_\mu + m]_{\alpha\beta}}{P^2 + m^2}$ ex. 9.2

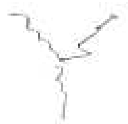
- Vertices = interactions:

quartic term $\quad \frac{1}{4} g^2 f^{abc} f^{ade} A_\mu^b A_\nu^c A_\mu^d A_\nu^e$

gluon

$$\overset{\text{ex 11}}{=} \frac{1}{4!} A_\mu^a A_\nu^b A_\rho^c A_\sigma^d\, g^2 \Big[f^{eab} f^{ecd} (\delta_{\mu\rho}\delta_{\nu\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})$$
$$+ f^{eac} f^{ebd} (\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\sigma}\delta_{\nu\rho})$$
$$+ f^{ead} f^{ebc} (\delta_{\mu\nu}\delta_{\rho\sigma} - \delta_{\mu\rho}\delta_{\nu\sigma})$$

Cubic terms:

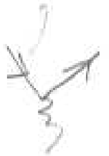


$$\frac{1}{2}\left[\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right] g f^{bca} A_\mu^b A_\nu^c$$

$$\overset{FT}{=} \int \frac{d^4P\, d^4Q\, d^4R}{(2\pi)^{12}} \delta^{(4)}(P+Q+R)\, \tilde{A}_\mu^a(P)\, \tilde{A}_\nu^b(Q)\, \tilde{A}_\rho^c(R)\; i g f^{abc}$$

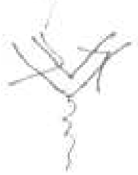
$$\left[\delta_{\mu\rho}(P_\nu - R_\nu) + \delta_{\rho\nu}(R_\mu - Q_\mu) + \delta_{\nu\mu}(Q_\rho - P_\rho)\right]$$

quark



$$\bar{\Psi}_{\alpha A}\, \gamma_{\mu\alpha\beta}\, (-i g A_\mu^a)\, T^a_{AB}\, \Psi_{\beta B}$$

ghost



$$\partial_\mu \bar{c}^a g f^{abc} A_\mu^b c^c \xrightarrow[FT]{} \int \frac{d^4P\, d^4Q\, d^4R}{(2\pi)^{12}} \tilde{\bar{c}}^a(P)\, \tilde{A}_\mu^b(Q)\, \tilde{c}^c(R)\; i g f^{abc} (-P_\mu)$$