5.4. Ward - Takahashi identity and self-energy in QED

[R] ch 7.4

- as for scalar field theory we can define a generating function for all Green's functions in QED:

normal. from FP

$$Z[\eta,\bar\eta,J] = N \int \mathcal{D}A_\mu \int \mathcal{D}\bar\psi \int \mathcal{D}\psi \;\exp\left[i \int d x^4 \mathcal{L}_{eff} \right] \text{, with: GF, sources}$$

$$\mathcal{L}_{eff} = \underbrace{-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\bar\psi\gamma^\mu D_\mu\psi - m\bar\psi\psi}_{\text{gauge invariant}} \;\; \underbrace{-\frac{1}{2\alpha}(\partial^\mu A_\mu)^2 + J^\mu A_\mu + \bar\eta\psi + \bar\psi\eta}_{\text{not gauge inv.}}$$

BUT: phys. quant. are expressed through Green's function $\Rightarrow$ $\boxed{\text{these \& } Z \text{ must be gauge inv}}$

infinitesimal gauge trafo:

$$\psi \to \psi' = e^{i\omega(x)}\psi = \psi + i\omega(x)\psi + \dots, \quad \bar\psi \to \bar\psi' = \bar\psi - i\omega(x)\bar\psi$$

$$A_\mu \to A_\mu' = A_\mu + \frac{1}{g}\partial_\mu\omega(x)$$

$\Rightarrow$ Z picks up a factor

$$\exp\left[i \int dx^4 \left(-\frac{1}{\alpha}\partial^\mu A_\mu \partial^\nu \frac{\partial_\nu\omega}{g} + J^\mu \frac{\partial_\mu\omega}{g} + i\bar\eta\omega\psi - i\omega\bar\psi\eta \right) \right]$$

$$\overset{\text{int. by parts}}{=} 1 + \frac{i}{g}\int dx^4 \left[-\frac{1}{\alpha}\Box\partial^\mu A_\mu - \partial_\mu J^\mu + ig(\bar\eta\psi - \bar\psi\eta) \right]\omega(x)$$

- we can generate the fields by functional derivatives w.r.t the sources

$$\psi(x) \to \frac{1}{i}\frac{\delta}{\delta\bar\eta(x)}, \quad \bar\psi(x) \to \frac{-1}{i}\frac{\delta}{\delta\eta(x)}, \quad A_\mu(x) \to \frac{1}{i}\frac{\delta}{\delta J^\mu(x)}$$

$$\Rightarrow \left[-\frac{1}{\alpha}\Box\partial^\mu\frac{\delta}{\delta J^\mu(x)} - i\partial_\mu J^\mu + ig\left(\bar\eta\frac{\delta}{\delta\bar\eta(x)} - \eta\frac{\delta}{\delta\eta(x)}\right) \right] Z[J,\bar\eta,\eta] = 0$$

Define $Z[\eta,\bar\eta,J_\mu] = \exp[iW[\eta,\eta,J_\mu]]$ for connected diagrams

and vertex function $\underline{\Gamma[\Psi,\bar\Psi,A_\mu] \equiv W[\eta,\bar\eta,J_\mu] - \int dx^4 (\bar\eta\Psi + \bar\Psi\eta + J^\mu A_\mu)}$

we have $-\frac{i}{\alpha}\Box\partial^\mu \frac{\delta W}{\delta J^\mu} - i\partial_\mu J^\mu - g\left(\eta\frac{\delta W}{\delta\bar\eta} - \eta\frac{\delta W}{\partial\eta}\right) = 0$

and
$$\boxed{\begin{array}{l} \frac{\delta\Gamma}{\delta A_\mu} = -J^\mu, \quad \frac{\delta\Gamma}{\delta\psi} = +\bar\eta, \quad \frac{\delta\Gamma}{\delta\bar\psi} = -\eta \\ \frac{\delta W}{\delta J^\mu} = A_\mu, \quad \frac{\delta W}{\delta\eta} = -\bar\Psi, \quad \frac{\delta W}{\delta\bar\eta} = \Psi \end{array}}$$

$$\stackrel{(\cdot i)}{\Rightarrow} \quad -\frac{1}{\alpha}\Box\partial^\mu A_\mu(x) + \partial_\mu\frac{\delta\Gamma}{\delta A_\mu(x)} - ig\left(\frac{\delta\Gamma}{\delta\psi}\psi(x) - \frac{\delta\Gamma}{\delta\bar\psi}\bar\psi(x)\right) = 0 \qquad (*)$$

Propagator and inverse ~:

- as before for the scalar field (ex 6.2, Euclidean) we have

$$\int dz \underbrace{\frac{\delta^2 W}{\delta\eta(x)\delta\bar\eta(z)}} \underbrace{\frac{\delta^2\Gamma}{\delta\Psi(z)\delta\bar\Psi(y)}} = \int dz \frac{\delta\Psi(z)}{\delta\eta(x)}\cdot\frac{\delta(-\eta(y))}{\delta\Psi(z)} = -\frac{\delta\eta(y)}{\delta\eta(x)} = -\delta^{(4)}(x-y)$$

$A \equiv 0$, $\Psi \equiv \bar\Psi = 0$ $\Rightarrow$ full propag. $\langle\bar\Psi(x)\Psi(z)\rangle = S_F'(x-z)$ inverse prop. $S_F'^{-1}$, after Fourier trafo:

$$\int dx \int dy\, e^{i(qx-py)} \frac{\delta^2\Gamma[0]}{\delta\bar\Psi(x)\delta\psi(y)} \equiv (2\pi)^4\delta(q-p)\, i\underline{S_F'^{-1}(p)}$$

- Ward - Takahashi :

differentiating $(*)$ twice wrt $\bar\Psi(x)$ and $\Psi(y)$ and setting $A=\Psi=\bar\Psi=0$ gives

$$-\frac{\partial}{\partial x^\mu}\frac{\delta^3\Gamma[0]}{\delta\bar\Psi(x_1)\delta\Psi(y)\delta A^\mu(x)} = -ig\,\delta^{(4)}(x-y)\frac{\delta^2\Gamma[0]}{\delta\bar\Psi(x_1)\delta\Psi(y)} + ig\,\delta^{(4)}(x-x_1)\frac{\delta^2\Gamma[0]}{\delta\bar\Psi(x_1)\delta\Psi(y)}$$

and after Fourier trafo

$$q^\mu \Gamma_\mu (p, q, p+q) = S_F^{\prime -1}(p+q) - S_F^{\prime -1}(p)$$

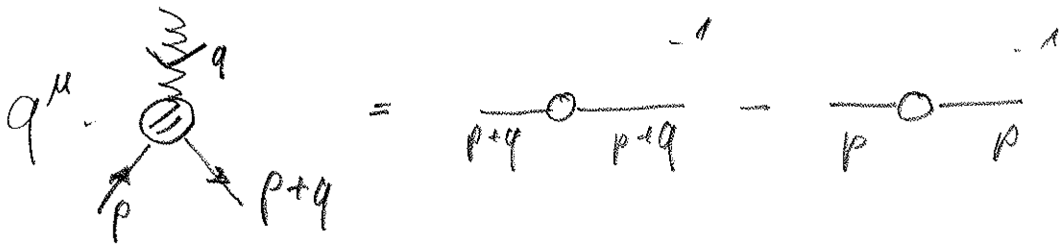
Ward-Takahashi id.

$$\lim_{q \to 0} : \quad \Gamma_\mu (p, 0, p) = \frac{\partial S_F^{\prime -1}}{\partial p^\mu}$$

Ward-id

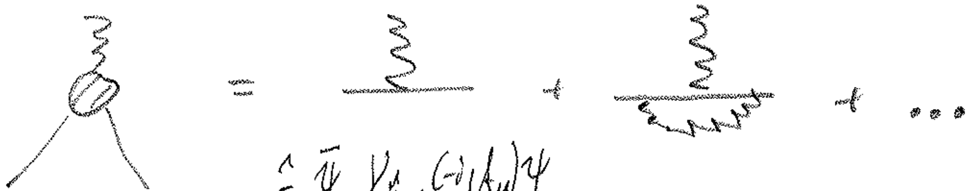
hold to all orders in perturbation theory

graphical representation:



∘ perturbative expansion:

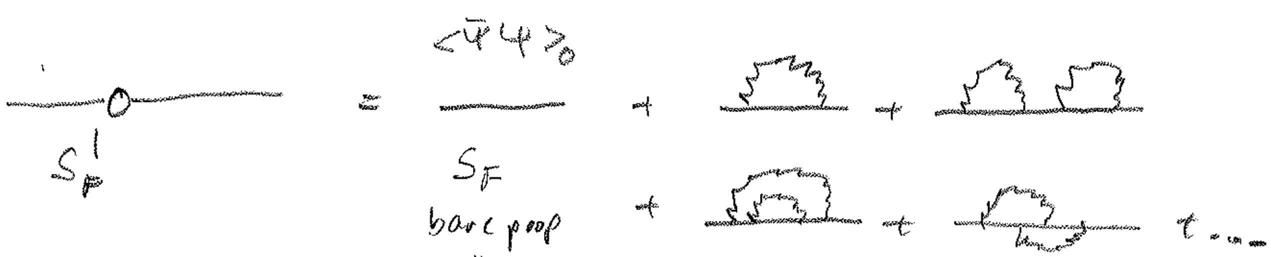
l.h.s



$\hat{=}\ \bar{\Psi}_\alpha \gamma^\mu_{\alpha\beta} (-g A_\mu) \Psi_\beta$

full 3-vertex

r.h.s expansion of el. propagator



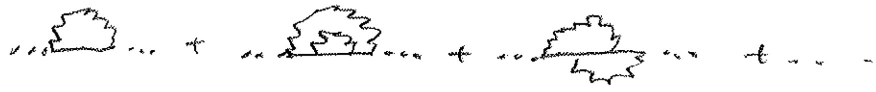
S_F bare prop

⇓

bare inv. prop. $S_F^{-1} = \gamma^\mu p_\mu - m$

• in analogy to the scalar field (p. 54) define 1PI

electron self energy Σ =



$$\Rightarrow \quad iS_F' = iS_F + iS_F \frac{\Sigma}{i} iS_F + iS_F \frac{\Sigma}{i} iS_F \frac{\Sigma}{i} iS_F + \ldots$$

$$= iS_F \frac{1}{1 - \Sigma S_F} = \frac{i}{S_F^{-1} - \Sigma}$$

$$\Leftrightarrow \quad S_F'^{-1} = S_F^{-1} - \Sigma = \gamma_\mu p^\mu - m - \Sigma(p)$$

$$\frac{\partial S_F'^{-1}}{\partial p^\mu} = \gamma_\mu - \frac{\partial \Sigma(p)}{\partial p^\mu}$$

$$\underset{\text{id}}{\overset{\text{Ward}}{\Rightarrow}} \quad \Gamma_\mu(p, q, p+q) \equiv \underset{\substack{\uparrow \\ \text{leading order}}}{\gamma_\mu} + \Lambda_\mu(p, q, p+q)$$

(check ex 11)

- the Ward-Takahashi identity is an important tool to prove the renormalizability of QED
- the NLO-diagram in S_F' (LO in Σ) is divergent  (as well as other diagrams)

$\rightarrow$ we have to regularise these in dimensional regularisation

Power counting for QED

- in analogy to scalar field theory we define

  D superficial degree of divergence, in d dimensions

  L # of loops, n # of vertices 

  P_i # internal photons, P_e # external photon lines

  E_i # " electrons, E_e # " electron " counting :

$$\underbrace{\tilde{\psi}\ \tilde{\bar{\psi}}} \sim \frac{-i\gamma_\mu^E p_\mu + m}{p^2 + m^2} \sim \frac{1}{|p|^1}$$

$$\underbrace{\tilde{A}\ \tilde{A}} \sim \frac{1}{p^2}\left(\delta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}\right) \sim \frac{1}{p^2}$$

 has momentum conservation

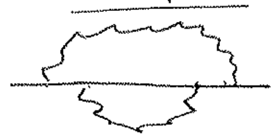
$$\Rightarrow \quad \boxed{D = dL - 2P_i - E_i}$$

- $L = E_i + P_i - n + 1$

\# indep p's for int. = # internal lines

overall mom. conserv.

example :



$L = 2,\ E_e = 2,\ P_e = 0$

$n = 4,\ E_i = 3,\ P_i = 2$

$\Rightarrow \ D = 1$

- per vertex :

has 2 electron lines $\quad 2n = E_e + 2E_i$

has 1 photon line $\quad n = P_e + 2P_i$

→ as before express D through n and external lines only :

$$D = d(E_i + P_i - n + 1) - 2P_i - E_i = E_i(d-1) + P_i(d-2) - d(n-1)$$

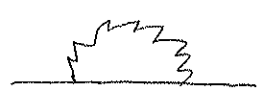
$$= \left(n - \tfrac{E_e}{2}\right)(d-1) + (n - P_e)\tfrac{1}{2}(d-2) - d(n-1)$$

$$\Leftrightarrow \quad D = d + n\left(\frac{d}{2} - 2\right) - E_e\frac{d-1}{2} - P_e\frac{(d-2)}{2}$$

$$\overset{d=4}{\Rightarrow} \quad \boxed{4 - \tfrac{3}{2}E_e - P_e = D}$$

n-indep !

- check with above example ok

-  has $D = 4 - \frac{3}{2}\cdot 2 = 1$