5.5. 1-loop renormalisation of QED

- we don't need the source terms to renormalise divergent diagrams

action in d dimensions:

$$\mathcal{L} = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi - eA^\mu\bar{\Psi}\gamma_\mu\Psi$$
$$-\frac{1}{4}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - \frac{1}{2}\left(\partial_\mu A^\mu\right)^2$$

with length dim $[S = \int d^dx\, \mathcal{L}] = \ell^0 \Rightarrow [\mathcal{L}] = \ell^{-d}$ $\quad ([x] = \ell)$

mass $[m] = \ell^{-1} \Rightarrow [\Psi] = [\bar{\Psi}] = \ell^{-\frac{d-1}{2}}$

$[\partial_\mu] = \ell^{-1} \Rightarrow [A_\mu] = \ell^{-(\frac{d}{2}-1)}$

- to get the dim of the coupling term right $[e]^\delta$ $\quad \Rightarrow \delta = 2 - \frac{d}{2}$

$$[eA^\mu\bar{\Psi}\gamma_\mu\Psi] = \ell^{\delta - (\frac{d}{2} - 1 + d - 1)} = \ell^{\delta - \frac{3}{2}d - 2} \stackrel{!}{=} \ell^{-d}$$

we need to introduce a dim-full parameter μ: $[\mu] = \ell$ (as in ϕ^4-theory)

$e \to e\mu^{2-\frac{d}{2}}$ at ✓ above

Electron self-energy:

L.O. $\quad -i\Sigma(p) = \left(-ie\mu^{2-\frac{d}{2}}\right)^2 \int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu}{k^2} \cdot \frac{1}{\not{p} - \not{k} - m}\gamma_\mu$

diverg. | D = 1 → dim reg

2 vertices $\qquad$ photon prop $\qquad$ el. prop

$$\Leftrightarrow \quad \Sigma(p) = -ie\mu^{4-d}\int \frac{d^dk}{(2\pi)^d} \frac{\gamma^\mu(\not{p} - \not{k} + m)\gamma_\mu}{\left((p-k)^2 - m^2\right)k^2}$$

Feynman parametrisation $\quad \frac{1}{ab} = \int_0^1 dz \frac{1}{[az + b(1-z)]^2}$

$$\Rightarrow \Sigma(p) = -ie^2\mu^{4-d}\int_0^1 dz \int \frac{d^dk}{(2\pi)^d} \frac{\gamma_\mu(\not{p}-\not{k}+m)\gamma^\mu}{\left((p-k)^2z - m^2z + k^2(1-z)\right)^2}$$

$k' = k - pz$, vmk i) p 53

$$\Sigma(p) = -ie^2\mu^{4-d}\int_0^1 dz \int \frac{d^dk'}{(2\pi)^d} \frac{\gamma_\mu(\not{p}(1-z) - \underline{\not{k}'} + m)\gamma^\mu}{\left(k'^2 - m^2z + p^2z(1-z)\right)^2}$$

vanishes when going to $d=4$ being odd $\not{k} = k_\mu\gamma^\mu$

$\Rightarrow$ we can do the k'-integral using ex 11.3

$$\int \frac{d^dk'}{(k'^2 + 2k'\cdot q - M^2)^A} = i(-\pi)^{\frac{d}{2}} \frac{\Gamma(A-\frac{d}{2})}{\Gamma(A)\left[-q^2-M^2\right]^{A-\frac{d}{2}}} \quad \text{with } A = 2, \; q = 0$$

$$-M^2 = -m^2z + p^2z(1-z)$$

$$\Rightarrow \Sigma(p) = \mu^{4-d}e^2\frac{\Gamma(2-\frac{d}{2})}{(4\pi)^{\frac{d}{2}}}\int_0^1 dz \frac{\gamma_\mu(\not{p}(1-z)+m)\gamma^\mu}{\left(m^2z - p^2z(1-z)\right)^{2-\frac{d}{2}}}$$

γ-matrices:

in d-dim we have (p 76, back to Minkowski):

- $\gamma^\mu\gamma_\mu = d \Rightarrow \gamma^\mu m\gamma_\mu = dm$
- $\gamma^\mu\not{p}\gamma_\mu = p^\nu\gamma^\mu\gamma_\nu\gamma_\mu = p^\nu\gamma^\mu(-\gamma_\mu\gamma_\nu + 2g_{\mu\nu}) = p^\nu\gamma_\nu\underline{(2-d)}$

setting $d = 4 - 2\varepsilon$ we have

$$\Sigma(p) = \mu^{2\varepsilon}\frac{e^2\Gamma(\varepsilon)}{(4\pi)^{2-\varepsilon}}\int_0^1 dz \frac{\not{p}(2\varepsilon-2)(1-z) + (4-2\varepsilon)m}{\left(m^2z - p^2z(1-z)\right)^{\varepsilon}}$$

where $\Gamma(\varepsilon) = \frac{1}{\varepsilon} - \gamma_{Euler} + O(\varepsilon)$

$$\Rightarrow \Sigma(p) = \frac{\mu^{2\varepsilon} e^2}{(4\pi)^{2-\varepsilon}} \left(\frac{1}{\varepsilon} - \gamma_{Euler} + O(\varepsilon)\right) \int_0^1 dz \frac{(4m - 2\not{p}(1-z)) + 2\varepsilon(\not{p}(1-z) - m)}{(m^2 z - p^2 z(1-z))^{\varepsilon}}$$

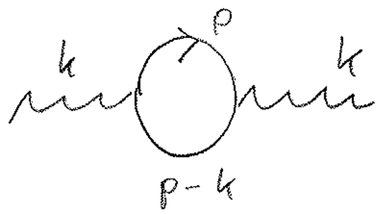
using $a^{-\varepsilon} = e^{-\varepsilon \ln a} = 1 - \varepsilon \ln a + O(\varepsilon^2)$ we get ($\int_0^1 dz = 1$, $\int_0^1 dz(1-z) = \frac{1}{2}$)

$$\boxed{\Sigma(p) = \frac{e^2}{16\pi^2}(4m - \not{p})\frac{1}{\varepsilon}} + O(1)$$

1 loop

. Below we shall only renormalise the $\frac{1}{\varepsilon}$-part (minimal subtraction)

- in analogy to the electron self-energy one can also def the

photon-self energy:  $\rightarrow \Pi_{\mu\nu}(k)$

w. ext. legs removed

From the Feynman rules we have

$$\Pi_{\mu\nu}(k) = i\mu^{4-d} e^2 \int \frac{d^d p}{(2\pi)^d} \mathrm{Tr}\left[\gamma_\mu \frac{1}{\not{p} - m} \gamma_\nu \frac{1}{\not{p} - \not{k} - m}\right]$$

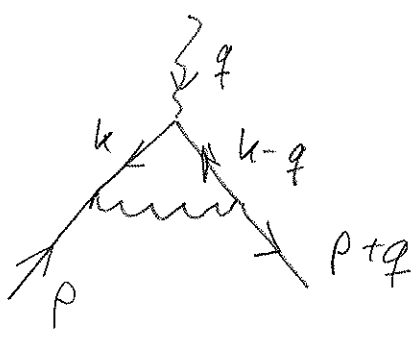
for the divergent part & 11.4 yields

$$\boxed{\Pi_{\mu\nu}(k) = \frac{e^2}{12\pi^2 \varepsilon}(k_\mu k_\nu - g_{\mu\nu} k^2)} + O(1)$$

1 loop

Vertex graph :

- there is a third, divergent graph, describing a 1 loop correction to the 3 vertex



In the Ward-identity (p. 94) we defined the loop correction as

$$\Gamma_\mu(p,q,p+q) = \gamma_\mu + \Lambda_\mu(p,q,p+q)$$

the Feynman rules give to 1 loop:

$$-ie\Lambda_\mu(p,q,p+q) = (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \frac{-ig_{\delta\sigma}}{(k+p)^2} \gamma^\delta \frac{i}{\not k - \not q - m} \gamma_\mu \frac{i}{\not k - m} \gamma^\sigma$$

a similar calculation in dim. regularisation yields (see [R], ch 9.5)

$2\varepsilon = \epsilon$

$$\boxed{\Lambda_\mu^{(1)}(p,q,p+q) = \frac{e^2}{16\pi^2}\gamma_\mu \cdot \frac{1}{\varepsilon}} + O(1)$$

1 loop

Ward-identity

as check we can see that to 1-loop the Ward-id is preserved (incl. dim-reg.!)

p. 93: $\boxed{\Gamma_\mu(p,0,p) = \frac{\partial S_F^{(-1)}}{\partial p^\mu}}$

with $S_F^{(-1)}(p) = S_F^{-1}(p) - \Sigma(p) = \gamma_\mu p^\mu - m - \Sigma(p)$

we have lhs $\gamma_\mu + \Lambda_\mu(p,0,p) = \gamma_\mu + \frac{e^2}{16\pi^2}\gamma_\mu \frac{1}{\varepsilon} + O(1)$

rhs $\partial_{p^\mu}\left(\gamma_\mu p^\mu - m - \frac{e^2}{16\pi^2}(4m - \not p)\frac{1}{\varepsilon} + O(1)\right)$

$= \gamma_\mu + \frac{e^2}{16\pi^2}\gamma_\mu \frac{1}{\varepsilon} + O(1)$ ✓

Renormalisation

as in ϕ^4-theory we call our parameters & fields in the initial $\mathcal{L}$ Bare (B)

$$\mathcal{L}_B = i\bar\psi_B \not\partial \psi_B - m_B\bar\psi_B\psi_B - e_B A^\mu_B \bar\psi_B\gamma_\mu\psi_B$$
$$- \frac{1}{4}(\partial_\mu A_{\nu B}-\partial_\nu A_{\mu B})(\partial^\mu A^\nu_B - \partial^\nu A^\mu_B) - \frac{1}{2\alpha_B}(\partial_\mu A^\mu_B)^2$$

Task: find factors
$$\boxed{Z_\psi^{\frac12}\psi_R=\psi_B,\quad Z_A^{\frac12}A^\mu_R = A^\mu_B \\ Z_m m_R = m_B,\quad Z_e e_R = e_B \\ Z_\alpha \alpha_R = \alpha_B}$$
s.th. the renormalized (R) quantities are finite

wave fnct. renorm.

- this is equivalent to adding counter terms to $\mathcal{L}_B$

$\Sigma(p)$: $\quad S_F^{(-1)}(p) = \not p\left(1+\frac{e^2}{16\pi^2\varepsilon}\right) - m\left(1+\frac{e^2}{4\pi^2\varepsilon}\right) + \dots \quad$ 2 different coeffs

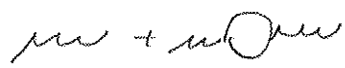
$\Rightarrow$ in $\mathcal{L}\big|_{\psi\,\text{only}} = iZ_\psi\bar\psi_R\not\partial\psi_R - Z_m Z_\psi m_R\bar\psi_R\psi_R$

$\hat=$ adding counter terms $\mathcal{L}_B\big|_\psi + iB\bar\psi\not\partial\psi - A\bar\psi\psi m$

we need $B = -\frac{e^2}{16\pi^2\varepsilon}$, $A = -\frac{e^2}{4\pi^2\varepsilon}$ $\Rightarrow$ $\boxed{Z_\psi = 1+B = 1-\frac{e^2}{16\pi^2\varepsilon}}$

$$\Rightarrow Z_m Z_\psi = 1+A \Leftrightarrow \boxed{Z_m} = \frac{\left(1-\frac{e^2}{4\pi^2\varepsilon}\right)}{\left(1-\frac{e^2}{16\pi^2\varepsilon}\right)} = \boxed{1-\frac{3e^2}{16\pi^2\varepsilon}} + O(e^4)$$

$\Pi_{\mu\nu}(p)$: $\quad D'_{\mu\nu}(k) = D_{\mu\nu}(k) + D_{\mu\alpha}(k)\Pi^{\alpha\beta}(k)D_{\beta\nu}(k) + \dots$



$$= -\frac{g_{\mu\nu}}{k^2}\left(1-\frac{e^2}{12\pi^2\varepsilon}\right) - \frac{e^2}{12\pi^2\varepsilon}\frac{1}{k^2}\frac{k_\mu k_\nu}{k^2} + \dots$$

corr. to α — not in Feynman gauge!

$\Rightarrow$ in $\mathcal{L}\big|_{A\,\text{only}} = -\frac{1}{4}Z_A(\partial_\mu A_{\nu R}-\partial_\nu A_{\mu R})^2 - \frac{Z_A}{Z_\alpha}\frac{1}{2\alpha_R}(\partial_\mu A^\mu_R)^2$

$\hat=$ adding counterterms $-\frac{C}{4}F_{\mu\nu}F^{\mu\nu} - \frac{C+E}{2}(\partial_\mu A^\mu)^2 + \mathcal{L}_B$

Lo. $D_{\mu\nu} = -\frac{1}{k^2}g_{\mu\nu} - \frac{\alpha-1}{(k^2)^2}k_\mu k_\nu$

$\frac{1}{2}A^\mu_B\Box A_{\mu B}$ at $\alpha=1$

$$\Rightarrow \quad C = + \frac{e^2}{12\pi^2\varepsilon} \quad \Rightarrow \quad \boxed{Z_A = 1 + C = 1 + \frac{e^2}{12\pi^2\varepsilon}}$$

correction to α: $\alpha - \frac{e^2}{12\pi^2\varepsilon} = \frac{1}{\xi} \Leftrightarrow \underset{\alpha=1}{E} = 1 + \frac{e^2}{12\pi^2\varepsilon}$, $\frac{Z_A}{Z_\alpha} = 1 + C + E$

$$\Leftrightarrow \quad Z_\alpha = \frac{Z_A}{1+C+E} = \frac{1+\frac{e^2}{12\pi^2\varepsilon}}{1+\frac{e^2}{6\pi^2\varepsilon}} = 1 - \frac{e^2}{12\pi^2\varepsilon} + O(e^4)$$

Vertex $\Gamma_\mu(p)$:

$$\Gamma(p, q, p+q) = \gamma_\mu \left(1 + \frac{e^2}{16\pi^2\varepsilon} \right) + \dots$$

$$\mathcal{L}_{\text{vertex}} = - Z_e Z_A^{\frac{1}{2}} Z_\psi e_R A^\mu_R \bar\psi_R \gamma_\mu \psi_R$$

$\hat{=}$ adding counterterm $- D e \bar\psi \not{A} \psi$, $D = - \frac{e^2}{16\pi^2\varepsilon}$

$$\Rightarrow \quad \boxed{Z_e = \frac{\left(1 - \frac{e^2}{16\pi^2\varepsilon}\right)}{\left(1 + \frac{1}{2}\frac{e^2}{12\pi^2\varepsilon}\right)\left(1 - \frac{e^2}{16\pi^2\varepsilon}\right)} = 1 - \frac{e^2}{24\pi^2\varepsilon}} + O(e^4)$$

<u>Conclusion</u>: QED is renormalisable to 1 loop

(given these were the only divergent diagrams)