The QCD-beta function and asymptotic freedom

Let us start with recalling the QCD Lagrangian in covariant gauge:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + i \sum_{f=1}^{n_f} \bar{\psi}_i^f [\not{D}]_{ij} \psi_j^f - \frac{1}{2\xi} (\partial^\mu A_\mu^a)^2 + \qquad (1)$$

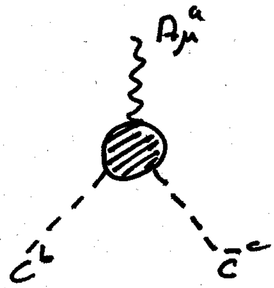
$$+ \partial^\mu \bar{c} (\partial_\mu c^a - g f^{abc} c^b A_\mu^c)$$

where $F^a_{\mu\nu} = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c$ and $[D_\mu]_{ij} = \delta_{ij} \partial_\mu - i g A_\mu^a T_{ij}^a$ with n_f massless quark flavours. From here we can compute several 2-point- and 3-point-functions.

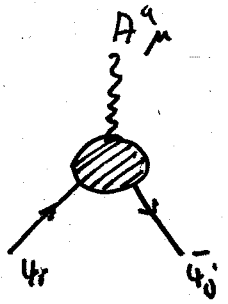
gluon propagator ($\equiv D^{ab}_{\mu\nu}$), $\underbrace{\ }_{\text{corresponding}}$ self-energy $\equiv \Pi(q^2)$

ghost propagator ($\equiv \Delta^{ab}$), — " — $\equiv \tilde{\Pi}(q^2)$

fermion propagator ($\equiv S_{ij}$), — " — $\equiv \Sigma_V(q^2)$



ghost-ghost-gluon vertex function $\equiv \tilde{\Gamma}(q^2)$



quark-quark-gluon vertex function $\equiv \Lambda(q^2), \Lambda^T(q^2)$

Lorentz structure is projected out

We use dimensional regularization in $D = 4-2\varepsilon$ dimensions and the minimal substraction renormal. scheme (MS). Then we have

$$\begin{aligned} (A_B)_\nu^a &= Z_3^{1/2}(h, \mu, \varepsilon)\, A_\nu^a(\mu), \\ c_B &= \tilde{Z}_3^{1/2}(h, \mu, \varepsilon)\, c(\mu), \\ (\psi_B)_i &= Z_2^{1/2}(h, \mu, \varepsilon)\, \psi_i(\mu), \\ h_B &= \mu^{2\varepsilon} Z_h(h, \mu, \varepsilon)\, h(\mu), \end{aligned} \qquad (2)$$

where $h = \frac{\alpha_s}{4\pi} = \frac{g_s^2}{(4\pi)^2}$ and μ being the renormalization parameter.

The renormalisation of self-energies and vertex function looks as follows

$$
\left.
\begin{aligned}
1+\Pi(q^2,h,\mu,\varepsilon) &= Z_3(h,\mu,\varepsilon)\left[1+\Pi_B(q^2,h,\varepsilon)\right]\Big|_{h_B=\mu^{2\varepsilon} Z_h\cdot h,\ \xi_B = Z_3\cdot\xi} \\
1+\tilde{\Pi}(q^2,h,\mu,\varepsilon) &= \tilde{Z}_3(h,\mu,\varepsilon)\left[1+\tilde{\Pi}_B(q^2,h,\varepsilon)\right]\Big|_{\dots} \\
1+\Sigma_V(q^2,h,\mu,\varepsilon) &= Z_2(h,\mu,\varepsilon)\left[1+(\Sigma_V)_B(q^2,h,\varepsilon)\right]\Big|_{\dots} \\
\tilde{\Gamma}(q^2,h,\mu,\varepsilon) &= \tilde{Z}_1(h,\mu,\varepsilon)\,\tilde{\Gamma}_B(q^2,h,\varepsilon)\Big|_{\dots} \\
\Lambda(q^2,h,\mu,\varepsilon) &= Z_1(h,\mu,\varepsilon)\,\Lambda_B(q^2,h,\varepsilon)\Big|_{\dots}
\end{aligned}
\right. \qquad (3)
$$

In this framework the renormalization constants Z_X can be written as a series of poles in ε

$$Z_X = 1+\sum_{i=1}^{\infty}\frac{1}{\varepsilon^i}Z_X^{(i)}\,,\quad Z_X^{(i)}(h)=\sum_{j\geq i} h^j Z_X^{(i,j)} \qquad (4)$$

and the μ-dependence is described as usual by the renormalization group equation

$$\mu^2\frac{\partial}{\partial\mu^2}h_B \overset{!}{=} 0 = \mu^{2\varepsilon}Z_h\Big[\varepsilon\cdot h + h\mu^2\frac{\partial\ln Z_h}{\partial\mu^2} + \underbrace{\mu^2\frac{\partial h}{\partial\mu^2}}_{\equiv\beta}\Big]$$

μ is arbitrary $\Rightarrow h_B$ independent of μ

$$\Rightarrow \beta = -\varepsilon\cdot h - h\mu^2\frac{\partial\ln Z_h}{\partial\mu^2} = -\varepsilon\cdot h - h\cdot\beta\frac{\partial}{\partial h}\ln Z_h \qquad \left(\mu^2\frac{\partial}{\partial\mu^2}=\mu^2\frac{\partial h}{\partial\mu^2}\frac{\partial}{\partial h}=\beta\frac{\partial}{\partial h}\right)$$

$$\Rightarrow \beta = -\frac{\varepsilon\cdot h}{1+h\partial_h\ln Z_h} = -\varepsilon\cdot h\left(1-h\partial_h\ln Z_h+\dots\right) = -\varepsilon\cdot h\left(1-h\partial_h\sum_{i=1}^{\infty}\frac{1}{\varepsilon^i}Z_h^{(i)}+\mathcal{O}(\hat{\varepsilon}^2)\right)$$

Demanding β to be finite in the limit $\varepsilon\to 0$ we get:

$$\boxed{\beta(\mu) = +h^2\partial_h Z_h^{(1)}} \qquad (5)$$

This means that the Beta function is determined by the $\frac{1}{\varepsilon}$ pole at any order in h. Back to the initial problem how to obtain the renormalization factor Z_h. There are several ways in computing Z_h. From Eqs. (2,3) we get immediatly

$$Z_h = \underbrace{Z_1^2 Z_2^{-2} Z_3^{-1}} = \underbrace{\tilde{Z}_1^2\tilde{Z}_3^{-2}Z_3^{-1}} = Z_{ggg}^2 Z_3^{-3} = Z_{gggg} Z_3^{-2}.$$

- qqg-vertex fct. + qq-propagator + gg-propagator → three independent calculations: (1973, Gross, Wilczek; 1973, Politzer; nobel-prize 2004)
- ghost-ghost-gluon vertex function + ghost-propagator + gluon-propagator → same here!
- renormalization constant for 3-gluon vertex function → two independent calculations (3g-vertex complicated)
- r.c. for 4-gluon vertex function → two calculations (complicated vertex function)

However, instead of calculating various vertex and propagator corrections, one can also choose a more suitable gauge namely background field gauge. As we know from page 106, the renormalization factor $Z_g = Z_h^{1/2}$ can be obtained directly by computing background field propagator corrections

$$Z_g^{\,2} = Z_B^{-1}. \qquad (6)$$

This means we reduce the calculation to one 2-point calculation instead of three independent 3- and 2-point corrections. However, the Feynman rules in bfg. are slightly more complicated. (Exercise: compute modified Feynman rules in background field gauge c.f. page 104-105)

Using Eq.(6), we get

$$\boxed{\beta(\mu) = -h^2 \partial_h Z_B^{(1)}}.$$

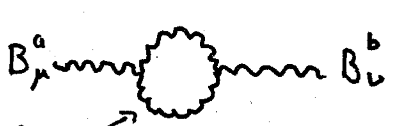
According to Eq.(4) we can write Z_B as

turns out to be zero.

$$Z_B(h) = 1 + \left(\frac{g}{4\pi}\right)^2 \frac{\beta_0}{\varepsilon} + \left(\frac{g}{4\pi}\right)^4 \left[\frac{\beta_1}{2\varepsilon} + \frac{\beta_{2,2}}{2\varepsilon^2}\right] + \mathcal{O}(g^6). \qquad (7)$$

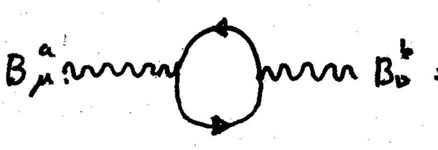
In the following we want to compute β_0. At 1-loop there are only three diagrams contributing to the background field propagator correction. In Feynman gauge ($\xi = 1$) we have

$$= -g^2 N_c \frac{\delta^{ab}\delta(P+Q)}{(P^2)^2} \int_k \frac{1}{k^2(P-k)^2}(2k_\mu - P_\mu)(2k_\nu - P_\nu)$$



$$= -\frac{g^2}{2} N_c \frac{\delta^{ab}\delta(P+Q)}{(P^2)^2} \int_k \frac{1}{k^2(p-k)^2}\Big[-8\cdot\delta_{\mu\nu}P^2 + (8-D)P_\mu P_\nu + 2D(K_\nu P_\mu - 2K_\mu K_\nu + K_\mu P_\nu)\Big]$$

internal field is gauge field A



$$= 2g^2 N_f \frac{\delta^{ab}\delta(P+Q)}{(P^2)^2} \int_K \frac{1}{K^2(P-K)^2}\Big[-8\delta_{\mu\nu}(K^2 - P\cdot K) + K_\nu P_\mu - 2K_\mu K_\nu + K_\mu P_\nu\Big]$$

with integration measure $\int_k = \int \frac{d^D k}{(2\pi)^D}$. Summing up all contributions and projecting out the transverse part (c.f. Exercise 12.2) we get

$$\Pi_B(P^2) = \frac{1}{D-1}\int_k \frac{1}{k^2(P-k)^2}\left[N_c\left(1 + \frac{7D-8}{2}\right) + N_f(2-D)\right] g_B^2 = \frac{1}{(4\pi)^2}\frac{1}{\varepsilon}\left[\frac{2}{3}N_f - \frac{11}{3}N_c\right] g_B^2 + \mathcal{O}(1)$$

According to Eq. (3) we have

$$\Pi = \overset{\text{background field}}{Z_B} \Big[1 + \underset{\text{bare}}{\Pi_B} \Big]_{g_B = \mu^{\varepsilon} Z_g \cdot g} - 1$$

$$= \left(\frac{g}{4\pi}\right)^2 \frac{1}{\varepsilon} \left[\frac{2}{3} N_f - \frac{11}{3} N_c \right] + \frac{g^2}{\varepsilon} Z_B^{(1,1)} + \frac{g^4}{\varepsilon^2} Z_B^{(1,1)} \left[\frac{2}{3} N_f - \frac{11}{3} N_c \right] + \mathcal{O}(1)$$

and therefore with Eq. 7

$$Z_B = 1 + \left(\frac{g}{4\pi}\right)^2 \left[\frac{11}{3} N_c - \frac{2}{3} N_f \right] \frac{1}{\varepsilon} + \mathcal{O}(g^4) \longrightarrow \boxed{\beta_0 = \frac{11}{3} N_c - \frac{2}{3} N_f}.$$

Solving the RG-Equation to leading order (c.f. ex. 7.4 or page 62) we get

$$g^2(\mu) \approx \frac{(4\pi)^2}{|\beta_0| \ln \frac{\mu}{\mu_0}}$$