6. Symmetries in QFT

- in <u>classical FT</u> we have considered the following

i) <u>variation of the fields</u> $\Rightarrow$ <u>Euler-Lagrange e.o.m</u> (see p. 3)

$$\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x) \qquad \boxed{0 = \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}}$$

ii) <u>internal symmetries</u> $\Rightarrow$ <u>Noether current</u> (p. 5)

$$x^\mu \to x'^\mu = x^\mu + X^\mu_i(x)\,\delta\omega^i$$
$$\phi^a(x) \to \phi'^a(x) = \phi^a(x) + \Phi^a_i(x)\,\delta\omega^i \quad \text{const.}$$
$$\Rightarrow \quad \partial_\mu j^\mu_i(x) = 0 \quad \text{with}$$
$$\boxed{j^\mu_i = \left[\mathcal{L}\delta^\mu_\nu - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)}\partial_\nu \phi^a\right] \overline{X(x)}_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a}\overline{\Phi(x)}^a_i}$$

iii) <u>local gauge symmetry:</u>

$$\phi'(x) = e^{i\omega(x)}\phi(x) = (1 + i\omega(x))\phi(x) + \mathcal{O}\omega^2$$

only a symmetry if we have $A'_\mu(x) = A_\mu(x) + g^{-1}\partial_\mu \omega(x)$ with <u>inhomogeneous transf. law</u>

Q: What happens to these concepts in QFT?

We have seen to i) operators (p. 11) or expect. val. (p. 43) satisfy e.o.m.

to ii) & iii) generating functionals satisfy sophisticated

- Schwinger–Dyson eqs for scalars (p. 44)
- Ward–Takahashi for QED (ψ & A_μ) (p. 92)

We will now take a more unified way of deriving these in the path integral formalism, especially in view of iii) and breaking of gauge inv. through gauge fixing ($\to$ BRST)

i) Equation of Motion — example scalar field theory

consider $Z = \int \mathcal{D}\phi \, \exp\left[i \int d^4x \, \mathcal{L}[\phi] \right]$, with $\mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{m^2}{2} \phi^2$

- in class. FT we would derive the Euler-Lagrange e.o.m from an infinitesimal change of var $\phi(x) \to \phi'(x) = \phi(x) + \delta\phi(x)$.
- ~~Here~~: we consider this as a change of var. in the path int

$$Z = \int \mathcal{D}\phi' \, e^{i \int d^4x \, \mathcal{L}[\phi']} = \int \mathcal{D}\phi \, e^{i \int d^4x \, \mathcal{L}[\phi']}$$

with inv. measure $\mathcal{D}\phi' = \mathcal{D}\phi$

$$= \int \mathcal{D}\phi \; i \int d^4x \, \delta\phi(x) \left[(-\Box_x - m^2) \phi(x) \right] e^{i \int d^4x \, \mathcal{L}[\phi]} \quad \text{to } \mathcal{O}.$$

$$= i \int d^4x \, \delta\phi(x) \, (-\Box_x - m^2) \int \mathcal{D}\phi \, \phi(x) \, e^{i \int d^4x \, \mathcal{L}}$$

$\forall$ var. $\delta\phi(x)$ so $\underline{(-\Box_x - m^2) \langle \phi(x) \rangle}$ $\qquad$ e.o.m. of expect. value (see p. 43)

- We can repeat the change of var. on higher expect. values / Green's funct:

$n=1$ $\quad \langle \phi(x_1) \rangle = \frac{1}{Z} \int \mathcal{D}\phi' \, e^{i \int d^4x \, \mathcal{L}[\phi']} \phi(x_1)' = \frac{1}{Z} \int \mathcal{D}\phi \, e^{i \int d^4x \, \mathcal{L}[\phi]} \phi(x_1)$

$$\Leftrightarrow \; 0 = \int \mathcal{D}\phi \, e^{i \int d^4x \, \mathcal{L}} \Big(i \int d^4x \, \delta\phi(x) \left[-\Box_x - m^2 \right] \phi(x) \, \phi(x_1) + \underline{\delta\phi(x_1)} \Big)$$

$\underline{\delta\phi(x_1)} \leftarrow -i \int d^4x \, \delta\phi(x) \, i \delta(x - x_1)$

$$\Leftrightarrow \; \boxed{ (\Box_x + m^2) \underbrace{\langle \phi(x) \phi(x_1) \rangle}_{} = i \delta(x - x_1) }$$

operator's Green's funct — or Klein-Gordon satisfied up to contact terms

n general

$$(\Box_x^2 + m^2) \langle \phi(x) \phi(x_1) \ldots \phi(x_n) \rangle$$

$$= \sum_{j=1}^{n} \langle \phi(x_1) \ldots (-i\delta(x - x_j)) \ldots \phi(x_n) \rangle$$

note: that we could have derived this using the Hamilton formalism and Wick's theorem

- for a more general scalar field theory with $\mathcal{L}$ we obtain

$$< \mathcal{L}[\phi]' > = 0 \quad ① \quad \text{exped. value of e.o.m.}$$

$$< \mathcal{L}[\phi]' \phi(x_1) \ldots \phi(x_n) > = \sum_{j=1}^{n} < \phi(x_1) \ldots i\delta(x-x_j) \ldots \phi(x_n) >$$

these are the Schwinger - Dyson eqs (which we derived in Euclid. metric with source term on p 44 and no extra fields)

ii) Noether current

recall for class. FT ($\overline{\mathcal{I}}^{\mu}_{i} \equiv 0$ for simplicity)

$$\Delta S' - \Delta S = \delta\omega^i \left[\frac{\partial S}{\partial \omega^i} = \int dx^4 \left\{ \frac{\partial \mathcal{L}}{\partial \phi^a} \frac{\partial \phi^a}{\partial \omega^i} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^a)} \frac{\partial \partial_\mu \phi^a}{\partial \omega^i} \right\}\right.$$

$$\stackrel{!}{=} \int dx^4 \left\{ \underbrace{\frac{\partial \mathcal{L}}{\partial \phi^a} \Phi_i^a - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} \Phi_i^a}_{\text{vanishes due to E.L. eom}} + \partial_\mu \Big[\underbrace{\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} \Phi_i^a}_{j^\mu_i} \Big] \right\}$$

QFT: • because we consider functional derivatives we need x-dep. trafo $\delta\omega^i(\underline{\underline{x}})$: $\phi^a_{(x)} \to \phi'^a_{(x)} = \phi^a_{(x)} + \Phi^a_{(x)i} \, \delta\omega^i(x)$

• assume an invariant measure $\mathcal{D}\phi^a = \mathcal{D}\phi^{a\prime}$

$$\Rightarrow \int \mathcal{D}\phi^{a\prime} \exp\{iS[\phi^{a\prime}]\} = \int \mathcal{D}\phi^a \exp\{iS[\phi^a]\}$$

$$\Rightarrow 0 = \int \mathcal{D}\phi^a \, i \underbrace{\frac{\delta S[\phi^a]}{\delta\omega^i(x)}}_{\text{p. 42} \; \hookrightarrow \frac{\partial \mathcal{L}}{\partial \omega^i(x)}} \exp\{iS[\phi^a]\}$$

$$\Leftrightarrow 0 = i \int \mathcal{D}\phi^a \left[\left(\frac{\partial \mathcal{L}}{\partial \phi^a} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} \right) \Phi_i^a(x) + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi^a} \Phi_i^a \right] \right] e^{iS}$$

$$\stackrel{①}{\Rightarrow} \boxed{ < \frac{\delta S[\phi^a]}{\delta\omega^i(x)} > = < -\partial_\mu j^\mu{}_i(x) > }$$

- the same identity can be formulated with more inserted operators observable $\mathcal{O}[\phi^a]$ (e.g. $\phi^a(x_1)\phi^a(x_2)\ldots$)

$$\int \mathcal{D}\phi^{a\prime}\, \mathcal{O}[\phi^{a\prime}]\, e^{iS[\phi^{a\prime}]} = \int \mathcal{D}\phi^a\, \mathcal{O}[\phi^{a}]\, e^{iS[\phi^a]}$$

$$\Leftrightarrow \int \mathcal{D}\phi^a \left\{ \frac{\delta \mathcal{O}[\phi^a]}{\delta\omega^i(x)} + i \frac{\delta S[\phi^a]}{\delta\omega^i(x)} \mathcal{O}[\phi^a] \right\} e^{iS[\phi^a]} = 0$$

$$\boxed{\left\langle \frac{\delta \mathcal{O}[\phi^a]}{\delta\omega^i(x)} \right\rangle = i \left\langle \frac{\delta S[\phi^a]}{\delta\omega^i(x)} \mathcal{O}[\phi^a] \right\rangle} \quad (*)$$

- these relations are also called Ward-Takahashi identities (we encountered these already but for generating functionals)

Example: 1) $\mathcal{L}_E = \bar{\Psi}[\gamma_\mu \partial_\mu + m]\Psi$

class FT: has a global rotation symmetry

$$\begin{aligned} \Psi \to \Psi' &= e^{i\omega}\Psi \\ \bar{\Psi} \to \bar{\Psi}' &= e^{-i\omega}\bar{\Psi} \end{aligned} \quad \Leftrightarrow \quad \begin{aligned} \delta\Psi &= i\Psi\delta\omega \\ \delta\bar{\Psi} &= -i\bar{\Psi}\delta\omega \end{aligned}, \quad \omega \in \mathbb{R} \text{ const}$$

class. Noether current $\frac{\partial \mathcal{L}_E}{\partial \partial_\mu \Psi} = \bar{\Psi}\gamma_\mu \Rightarrow j_\mu = \frac{\partial \mathcal{L}_E}{\partial \partial_\mu \Psi} \cdot i\Psi = i\bar{\Psi}\gamma_\mu\Psi$ is conserved

QFT: $S_E = \int dy^4\, \bar{\Psi}(y)[\gamma_\mu \partial_\mu + m]\Psi(y)$, trafo $\delta\Psi(x) = i\Psi(x)\delta\omega(x)$

$$S_E[\Psi'] - S_E[\Psi] = \int dy^4 \left\{ -i\bar{\Psi}(y)\delta\omega(y)[\gamma_\mu\partial_\mu + m]\Psi(y) + i\bar{\Psi}(y)[\gamma_\mu\partial_\mu + m](\delta\omega(y)\Psi(y)) \right\}$$

$$= \int d^4y \left\{ i\bar{\Psi}(y)\gamma_\mu(\partial_\mu\omega(y))\Psi(y) \right\}$$

$$\Rightarrow \frac{\delta S_E[\Psi']}{\delta\omega(x)} = -i\partial_\mu(\bar{\Psi}(x)\gamma_\mu\Psi(x)) = -\partial_\mu j_\mu(x)$$

so $\langle \partial_\mu j_\mu(x) \rangle = 0$

Example 2) QCD with N_f flavours $(u, d, s, \dots)$

F^a $SU(N_f)$ generators, $[F^a, F^b] = i h^{abc} F^c$

$$\psi = \begin{pmatrix} \psi_u \\ \psi_d \\ \psi_s \\ \vdots \end{pmatrix} \quad \text{4 spinor}$$

$\exists$ 2 types of global rotations:

$$\delta_V^a \psi = i F^a \psi \, \delta\omega_V \;, \quad \delta_V^a \bar\psi = -i \bar\psi F^a \delta\omega_V \qquad \text{vector (V) trafo}$$

$$\delta_A^a \psi = i F^a \gamma_5 \psi \, \delta\omega_A \;, \quad \delta_A^a \bar\psi = +i \bar\psi \gamma_5 F^a \delta\omega_A \qquad \text{axial vector (A) trafo}$$

in analogy to example 1) we get for the QFT trafo on the action

$$\delta_V^a S_E = \int d^4y \, \delta\omega_V(y) \, (-)\partial_\mu \left(i\bar\psi(y) \gamma_\mu F^a \psi(y) \right) \Rightarrow \boxed{\frac{\delta S}{\delta\omega_V^a} = -\partial_\mu i \bar\psi \gamma_\mu F^a \psi}$$

$$\delta_A^a S_E = \int d^4y \left\{ i\bar\psi \gamma_5 F^a \delta\omega_A(y) \left[\gamma_\mu \partial_\mu + m \right] \psi + \bar\psi \left[\partial_\mu \gamma_\mu + m \right] i F^a \gamma_5 \delta\omega_A(y) \psi \right\}$$

$$0 = \{\gamma_\mu \gamma_5\} \searrow \; = \int d^4y \left\{ i\bar\psi F^a \gamma_5 \gamma_\mu \psi \, \partial_\mu \left(\delta\omega_A(y) \right) + 2mi \bar\psi F^a \gamma_5 \psi \, \delta\omega_A(y) \right\}$$

$$\Rightarrow \boxed{\frac{\delta S_E}{\delta\omega_A^a} = -i \partial_\mu \left(\bar\psi F^a \gamma_5 \gamma_\mu \psi \right) + 2mi \bar\psi F^a \gamma_5 \psi}$$

conserved current only for massless quarks

- if we insert operators $\mathcal{O}$ such as

$$A_\mu^a(x) \equiv i\bar\psi(x) F^a \gamma_\mu \gamma_5 \psi(x) \;, \quad P^a(x) \equiv i\bar\psi(x) F^a \gamma_5 \psi(x)$$

one can derive so-called current algebra relations

$$0 = \frac{\partial}{\partial x^\mu} \langle A_\mu^a(x) P^b(y) \rangle + 2m \langle P^a(x) P^b(y) \rangle$$

<u>P</u>artially <u>C</u>onserved <u>A</u>xial <u>C</u>urrent

- For <u>Ward identities with source terms</u> we simply insert $\mathcal{O} = e^{\int d^4x \, j^a(x) \phi^a(x)}$ in (*)

$\rightarrow$ as we have seen these are important in the renormalisability of our theory