6.2 BRST - Symmetry

[Becchi, Rouet, Stora, Comm. Math. Phys 42 (1975) 127, I.V. Tyutin, unpublished]

- we will construct a generalised "gauge trafo" with fermionic trafo parameter that leaves the Yang-Mills action + gauge fixing (GF) + FP term invariant

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} - \frac{1}{2\xi} G^a G^a - c^{*a} \frac{\partial G^a}{\partial \theta^b} c^b \qquad \text{(cf. p 87)}$$

a standard infinitesimal gauge trafo $(U = e^{igT^a\theta^a(x)})$ would read

$$A'^a_\mu = A^a_\mu + \underbrace{\left(\partial_\mu \delta^{ab} - g f^{abc} A^c_\mu\right)}_{\equiv D^{ab}_\mu = \frac{\partial A^a_\mu}{\partial \theta^b}} \theta^b = A^a_\mu + \delta A^a_\mu$$, it leaves $\frac{1}{4}F^2$ inv, not the rest

consider the set of trafo's with $\theta^b \to -c^a\lambda$, c^a, λ Grassmann Va

BRST trafo

$$\boxed{\begin{aligned} sA^a_\mu &= -D^{ab}_\mu c^b \lambda \\ sc^a &= -\tfrac{1}{2} g f^{abc} c^b c^c \lambda \\ sc^{*a} &= -\frac{1}{\xi} G^a \lambda \end{aligned}}$$

being of the form of a gauge trafo this leaves the gauge part invariant

- we shall show now that $s(\mathcal{L}_{GF} + \mathcal{L}_{FP}) = 0$

$$s\left(-\frac{1}{2\xi} G^a G^a\right) = -\frac{1}{\xi} G^a s G^a = -\frac{1}{\xi} G^a \frac{\partial G^a}{\partial A^b_\mu} s A^b_\mu = +\frac{1}{\xi} G^a \frac{\partial G^a}{\partial A^b_\mu} \overbrace{D^{bc}_\mu}^{\frac{\partial A^b_\mu}{\partial \theta^c}} c^c \lambda$$

$$= +\frac{1}{\xi} G^a \frac{\partial G^a}{\partial \theta^c} c^c \lambda$$ for general G^a

$$s\left(-c^{*a} \frac{\partial G^a}{\partial \theta^b} c^b\right) = +\frac{1}{\xi} G^a \lambda \frac{\partial G^a}{\partial \theta^b} c^b - c^{*a} s\left(\frac{\partial G^a}{\partial \theta^b} c^b\right)$$

gives −

- so if we can show

$$0 = s\left(\frac{\partial G^a}{\partial \theta^b} c^b\right)$$ the full action $\mathcal{L}$ is invariant.

$$s\left(\frac{\partial G^a}{\partial \theta^b} c^b\right) = s\left(\frac{\partial G^a}{\partial A_\mu^d}\frac{\partial A_\mu^d}{d\theta^b} c^b\right) = s\left(\frac{\partial G^a}{\partial A_\mu^d} \mathcal{D}_\mu^{db} c^b\right)$$

$$= \underbrace{\frac{\partial^2 G^a}{\partial A_\mu^d \partial A_\nu^e}}\,\underbrace{s A_\nu^e \mathcal{D}_\mu^{db} c^b} + \frac{\partial G^a}{\partial A_\mu^d}\, s\left(\mathcal{D}^{db} c^b\right) = 0$$

as we have $= (\mathcal{D}_\nu^{ef} c^f)\lambda(\mathcal{D}_\mu^{db} c^b) = -(\mathcal{D}_\mu^{db} c^b)\lambda(\mathcal{D}_\nu^{ef} c^f)$ antisym

whereas is symm in exchanging $e \leftrightarrow d$, $\mu \leftrightarrow \nu$ $\rightsquigarrow$ gives 0

- together with $\boxed{s(\mathcal{D}^{db} c^b) = 0}$ from ex 13.3

this gives the desired result for general GF choice of G^a

Note that this automatically implies

$$s^2 A_\mu^a = -s(\mathcal{D}_\mu^{ab} c^b)\lambda = 0$$ without using the nil-potency of $\lambda^2 = 0$

the same is true for $s^2 c^a = 0$ due to $s(-\frac{1}{2} g f^{abc} c^b c^c) = 0$ (ex 13

Slavnov - Taylor identities:

we are now in the position to derive identities valid for non-Abelian gauge theory and their gen. functions with sources. These are important to prove their renormalisability

$$Z[t, x, y; u, v] \equiv \int \mathcal{D}A_\mu \mathcal{D}\bar{c}^* \mathcal{D}c \, \exp\left[i \int \mathcal{I}_{tot} d^4x\right]$$, with

$$\mathcal{I}_{tot} = \mathcal{L} + t_\mu^a A^{a\mu} + x^a c^a + y^a c^{*a} + u_\mu^a (\mathcal{D}_\mu^{ab} c^b) + v^a\left(-\tfrac{g}{2} f^{abc} c^b c^c\right)$$

with standard sources (t, x, y) and extra sources to linearise (u, v)

- we have just seen that $s\mathcal{I} = 0$, as well as the invariance of the u- and v- term.

- as in the derivation of the Ward-Takahashi id. (p. 91) we can then impose the invariance of $Z[t, x, y; u, v]$ leading to the desired identity.

But first we need to compute the Jacobian of our s-trafo:

$$J = \det\left[\frac{\partial\, A_\mu^a + sA_\mu^a,\ c^a + sc^a,\ c^{*a} + sc^{*a}}{\partial\, A_\mu^a,\ c^a,\ c^{*a}}\right]$$

$$sA_\mu^a = -(\partial_\mu c^a + g f^{abc} A_\mu^b c^c)\lambda$$

$$sc^a = -\frac{g}{2} f^{abc} c^b c^c \lambda$$

$$sc^{*a} = -\frac{1}{\xi}\partial_\mu A_\mu^a \lambda \quad \text{choosing the standard } G^a$$

so $\frac{\delta(A_\mu^a + sA_\mu^a)(x)}{\delta A_\nu^b(y)} = \delta_\mu^\nu \delta(x-y)\,(\delta^{ab} + g f^{acb} c^c)$ etc leading to

$$J = \delta_\mu^\nu \delta(x-y) \begin{vmatrix} 1 - gf\lambda & +gfA & 0 \\ 0 & 1 + gfc\lambda & 0 \\ -\frac{\lambda}{\xi}\partial & 0 & 1 \end{vmatrix} \hat{=} 1$$

- with $S = \int d^4x\, \mathcal{L}_{tot}$ we get upon expanding in Z:

$$e^{\int d^4x (t_\mu^a sA_\mu^a + x^a sc^a + y^a sc^{*a})} = 1 + \int d^4x\, [t_\mu^a sA_\mu^a + x^a sc^a + y^a sc^{*a}]$$

$$\Rightarrow \int \mathcal{D}A_\mu \int \mathcal{D}c^{*a} \int \mathcal{D}c^a \int d^4x \left[-t_\mu^a (D_\mu^{ab} c^b)\lambda - x^a \underbrace{\left(\frac{g}{2} f^{abc} c^b c^c\right)}\lambda - y^a \frac{1}{\xi}\underbrace{\partial_\mu A_\mu^a}\lambda\right] e^{iS}$$

- these derivatives can be pulled out of the integral, and upon def $Z = \exp[iW[t, x, y; u, v]]$ we have

$$\frac{\delta S}{\delta u_\mu^a} \qquad \frac{\delta S}{\delta v^a} \qquad \partial_\mu \frac{\delta S}{\delta t_\mu^a}$$

$$\int d^4x \left[t_\mu^a \frac{\delta W}{\delta u_\mu^a} + x^a \frac{\delta W}{\delta v^a} + \frac{y^a}{\xi} \partial_\mu \frac{\delta W}{\delta t_\mu^a} \right] = 0$$

Finally we define a Legendre trafo w.r.t. sources t, X, Y:

$$W[t, X, Y; u, \sigma] = \Gamma[A, c, c^*; u, \sigma] + \int d^4x \left(t_\mu^a A_\mu^a(x) + X^a c^a(x) + Y^a c^{*a}(x)\right)$$

and thus $\frac{\delta W}{\delta u} = \frac{\delta \Gamma}{\delta u}$, $\frac{\delta W}{\delta \sigma} = \frac{\delta \Gamma}{\delta \sigma}$ remain unchanged

whereas $t_\mu^a = -\frac{\delta \Gamma}{\delta A_\mu^a}$, $X^a = -\frac{\delta \Gamma}{\delta c^a}$, $Y^a = -\frac{\delta \Gamma}{\delta c^{*a}}$

$$\Rightarrow \boxed{\int d^4x \left[\frac{\delta \Gamma}{\delta A_\mu^a}\frac{\delta \Gamma}{\delta u_\mu^a} + \frac{\delta \Gamma}{\delta c^a}\frac{\delta \Gamma}{\delta \sigma_a} + \frac{1}{\xi}(\partial_\mu A_\mu^a)\frac{\delta \Gamma}{\delta c^{*a}}\right] = 0}$$

(can be absorbed by $\Gamma' = \Gamma + \frac{1}{\xi}\int d^4x (\partial_\mu A_\mu^a)^2$ (see [R ch 7.6] for details)

these identities are called

Slavnov - Taylor identities

- the BRST - formalism allows an elegant classification of the physical subset of gauge - inv. states
  - linearisation: $\exp\left(-\frac{1}{2\xi} G^a G^a\right) \sim \int_{-\infty}^{\infty} dB \exp\left[-\frac{\xi}{2} B^2 - i B G\right]$

    $\Rightarrow I = I[\phi]$, ϕ all fields $\{A_\mu^a, c^{*a}, c^a, B^a\}$ $\quad (G^a = \partial_\mu A_\mu^a)$

    with $sI = 0$, $\quad s\phi = \{D_\mu^{ab} c^b, iB^a, \frac{g}{2} f^{abc} c^b c^c, \overset{\text{def}}{0}\}$, $s^2\phi = 0$

- all states are of the form:

  $|\psi_1\rangle$ states with $s|\psi_1\rangle \neq 0$ — unphys., not inv.

  $|\psi_2\rangle$ states with $|\psi_2\rangle = s|\psi_1\rangle \Rightarrow s|\psi_2\rangle = 0$, unphys as trafo of something ("pure gauge")

  $|\psi_0\rangle$ states with $|\psi_0\rangle \neq s|\psi_1\rangle$ but $s|\psi_0\rangle = 0$ — physical states, inv.

- this classification is completely analogous to forms in differential geometry $\quad e^{\lambda s}|\psi_0\rangle = |\psi_0\rangle$