[BRST literature in new arXiv 1107.1070]

7. The large N_c - limit (review A.V. Manohar, hep-ph/9802419)

- It goes back to the idea of Gerard 't Hooft that QCD simplifies in the limit $N_c \to \infty$ when computing Feynman diagrams
- in many cases the corrections to $N_c = \infty$ are $\mathcal{O}\left(\frac{1}{N_c^2}\right)$, $\sim 10\%$ for $N_c = 3$
- we shall see in some toy model how far the computations at $N_c = \infty$ can be made explicit

1) The Gross-Neveu Model: interacting Fermions in $d = 1+1 = 2$

$$\boxed{\mathcal{L} = \bar\Psi i\partial\!\!\!/ \Psi + \frac{\lambda}{2}(\bar\Psi\Psi)^2}$$

, Ψ^a $a = 1,\dots,N$ 2-spinors

$\bar\Psi\Psi = \bar\Psi^a\Psi^a$

- dimensions: $[\Psi] = \ell^{-\frac{1}{2}}$ $[\partial_\mu] = \ell^{-1}$ $\Rightarrow [S = \int d^2x\,\mathcal{L}] = \ell^0$, $[\lambda] = \ell^0$
- is renormalisable:

degree of divergence D: modify ϕ-theory p. 49

$$D = d + \underbrace{N}_{\text{vertices}}(d-2) - (d-1)\underbrace{\frac{E}{2}}_{\text{ext. leg's}} \underset{d=2}{=} 2 - \frac{1}{2}E \quad \checkmark$$

Symmetries: $\Psi^a \to U^a{}_b \Psi^b$, $U \in SU(N)$

& discrete $\Psi \to \gamma_5 \Psi$, $\bar\Psi \to -\bar\Psi\gamma_5$, $\bar\Psi\Psi \to -\bar\Psi\Psi$, $\bar\Psi\gamma_\mu\partial_\mu\Psi$ inv

$\Rightarrow$ no mass term, $\forall$ order in pert. theory $\qquad \{\gamma_\mu, \gamma_5\} = 0$

diagrams:

of order λ, λ^2, $\lambda^2 \cdot N$, $\lambda^3 N^2$ etc

$\frac{g^2}{N}$, $\frac{g^4}{N^2}$, $\frac{g^4}{N}$ (sum over all internal c), $\frac{g^6}{N}$

$\Rightarrow$ perturbation theory cannot make sense when $N \to \infty$ without rescaling $\lambda = \lambda(N)$: $\boxed{\lambda = \frac{g^2}{N}}$

this can also be seen from β-funct.

• $\frac{\bar\psi\psi}{\sqrt{N}}$ is the proper normalisation of N-part. state $|\psi\rangle \sim (|1\rangle + .. |N\rangle)$

$$\mathcal{L} = \bar\psi i \partial\!\!\!/ \psi + \frac{g^2}{2N}(\bar\psi\psi)^2 \underset{\uparrow}{=} \bar\psi i \partial\!\!\!/ \psi + \sigma\bar\psi\psi - \frac{N}{2g^2}\sigma^2$$

intro non dyn. aux field σ, min $\mathcal{L} \to \sigma = \frac{g^2}{N}\bar\psi\psi$

new Feynman rules:

ψ:  $\frac{i}{p\!\!\!/}$, σ:  $-\frac{ig^2}{N}$ (const. prop.)  1

- counting for diagrams still complicated: implicit N from ψ-loops

→ effective theory for σ:

$$Z = \int \mathcal{D}\bar\psi\mathcal{D}\psi\int\mathcal{D}\sigma\, e^{i\int d^4x\, \mathcal{L}(\psi,\sigma,g,N)} \equiv \int\mathcal{D}\sigma\, e^{i\int d^4x\, \mathcal{L}_{eff}(\sigma,g,N)} =$$

at 1 loop level one can see that $\boxed{\mathcal{L}_{eff} = N\hat{\mathcal{L}}_{eff}(\sigma,g)}$

from  N from ψ-loop

so σ prop $\sim \frac{1}{N}$, all vertices $\sim N$

easy counting

• due to this rescaling the limit $N\to\infty$ reminds of the semiclassical limit for the aux. field σ (more details: A. Manohar)

2) QCD:

p (110)

Beta function $\beta(\mu) = \mu\frac{dg}{d\mu} = -\left(\frac{11}{3}N_c - \frac{2}{3}N_f\right)\frac{g^3}{16\pi^2} + \mathcal{O}(g^5)$

⇒ to obtain a well-defined limit we have to rescale (as before)

$$g \to \frac{\hat g}{\sqrt{N_c}} \Rightarrow \boxed{\mu\frac{d\hat g}{d\mu} = -\left(\frac{11}{3} - \frac{2}{3}\frac{N_f}{N_c}\right)\frac{\hat g^3}{16\pi^2} + \mathcal{O}\left(\frac{\hat g^5}{N^2}\right)}$$

• at large $N_c\to\infty$ the effect of Fermion-loops gets suppressed (unless we keep $\frac{N_f}{N_c}$ fixed)

• ∃ certain (susy) YM theorys $SU(N_c)$ that become 1-loop exact

- after rescaling the Lagrangian contains $\hat{g}$ as

$$D_\mu = \partial_\mu - i\frac{\hat{g}}{\sqrt{N}}A_\mu \;,\; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\frac{\hat{g}}{\sqrt{N}}[A_\mu, A_\nu]$$

- in order to make the counting easier we further rescale $\frac{\hat{g}}{\sqrt{N}}A_\mu \to \hat{A}_\mu$, $\psi \to \sqrt{N}\hat{\psi}$, $c \to \sqrt{N}\hat{c}$

$$\Rightarrow \mathcal{L} = N\hat{\mathcal{L}} \;,\; \hat{\mathcal{L}} = -\frac{1}{4g^2}\hat{F}^a_{\mu\nu}\hat{F}^{a\,\mu\nu} + i\bar{\psi}\not{D}(\hat{A})\psi - \frac{1}{2\xi}(\partial_\mu \hat{A}^a_\mu)^2 + \partial^\mu \bar{c}\, D_{\mu b}^a(\hat{A})\, c^b$$

* each vertex $\sim N$, each propagator $\sim \frac{1}{N}$ (we will focus on $\hat{A}$, $\hat{\psi}$ below)
(unlike in GN this $\neq$ semiclass limit, # of comp of A_μ, ψ still N-dep)

Counting rules

using a trick due to 't Hooft we use matrix rep for A^a_μ & use double lines:

$$(A_\mu)^a{}_b = A^A_\mu (T^A)^a{}_b \qquad , (T^A)^a{}_b \text{ matrix rep of } SU(N_c)$$

$$\Rightarrow \langle \hat{A}^A_\mu(x)\hat{A}^B_\nu(y)\rangle = \frac{g^2}{N}\delta^{AB} D_{\mu\nu}(x-y) \qquad | \cdot T^A\, T^B$$

$$\Rightarrow \boxed{\langle (\hat{A}_\mu(x))^a{}_b\, (\hat{A}_\nu(y))^c{}_d\rangle = \frac{g^2}{N} D_{\mu\nu}(x-y)\left(\frac{1}{2}\delta^a_d\delta^c_b - \frac{1}{2N}\delta^a_b\delta^c_d\right)}$$

using $\sum_A (T^A)^a{}_b (T^A)^c{}_d = \frac{1}{2}\left(\delta^a_d\delta^c_b - \frac{1}{2N}\delta^a_b\delta^c_d\right)$ for $T\in SU(N_c)$

compare to $\qquad = \frac{1}{2}\delta^a_d\delta^c_b$ for $T \in U(N_c)$

" $SU(N_c) = U(N_c)$ " at large-N_c

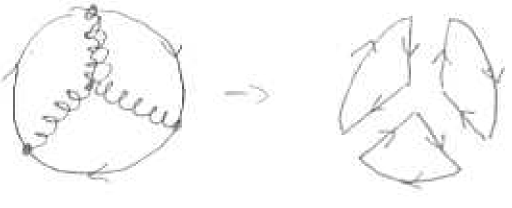
$\rightsquigarrow$ will consider $U(N_c)$ in the following (the extra $U(1)$ ($SU(N_c) = U(N_c)$) can be cancelled with an extra ghost

$\langle \hat{\psi}^a(x)\bar{\hat{\psi}}^b(y)\rangle = \delta^{ab}S(x-y)$ is a scalar and remains a single line

• each colour loop gives a factor of N

examples for diagrams (without external legs)

1)

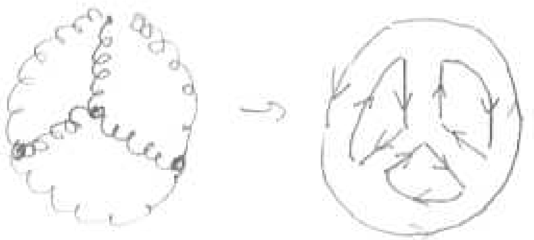


order of diagram: 3 loops, 6 prop., 4 vertices

$$N^3 N^{-6} N^4 = N$$

((old counting for $\mathcal{L}$ before rescaling: $N^3 \cdot 1 \cdot \left(\frac{1}{\sqrt{N}}\right)^4 = N$ ✓)

2)

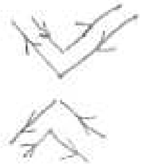


+1 loop => $N^{1+1} = N^2$

A few important facts

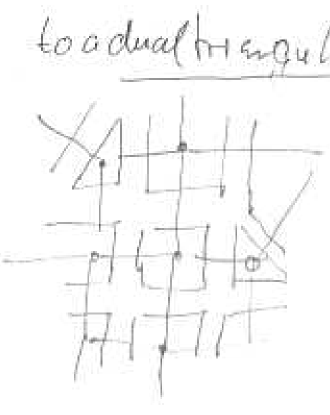
• 3- and 4-gluon vertex come from a single trace $\mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu}$

⇝ can only contract colour lines from outside, not inside a vertex, so no diagr.



• consider only diagr with gluons (N_f suppressed in $\beta(\mu)$, or here: quark loops give suppression $\frac{1}{N}$ = boundary, b example 1) vs 2))

- the double-line graphs corresp. to a dual triangulation of orientable surfaces



with degree

Euler characteristic

$$N^{L-P+V} = N^{\chi}, \quad \chi = 2 - 2h \ (-b)$$

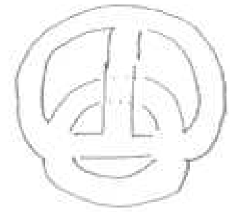
with genus h = 0: sphere, 1: torus, ...

$\Rightarrow$ planar diagrams with only (internal) gluon lines dominate

(that can be drawn on a plane $\simeq$ sphere)

examples:

planar,



non-planar

$N^{4} N^{-4} N^{+2} = N^{2}$ $\qquad$ $N^{3} N^{-6} N^{3} = N^{0}$

- the interpretation as triangulations of surfaces has a formal equivalent to that in string theory

- a disconnected diagram with c connected parts can be at most of order $(N^2)^c$ $\rightarrow$ if we consider only connected diagrams we have $\underline{Z = e^{N^2 W(g,N)}}$, W generating function with $\lim_{N\to\infty} W = O(1)$

- In certain toy models (in dim 0 or 1) the sum over all planar diagrams can be explicitly computed: "Planar diagrams" Brézin, Itzykson, Parisi, Zuber Com. Math Phys 59 (1978) 35-51

$\rightarrow$ matrix models

- in QCD we of course still have to do take into account p-integrals with propagators $D_{\mu\nu}(x-y)$ etc.

- the large-N limit is also useful in effective field theories of QCD (e.g. chiral perturbation theory) and in phenomenological considerations for mesons & baryons