

[Wednesday 13. April 2011, 12:30-14:00 in D5-153]

Exercise 2.1: Consider the harmonic oscillator with

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2 \hat{x}^2}{2}, \quad \hat{a} = \sqrt{\frac{m\omega}{2}} \hat{x} + \frac{i}{\sqrt{2m\omega}} \hat{p}, \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2}} \hat{x} - \frac{i}{\sqrt{2m\omega}} \hat{p}.$$

Starting from $[\hat{x}, \hat{p}] = i$ and $[\hat{x}, \hat{x}] = [\hat{p}, \hat{p}] = 0$, show that:

- (a) $[\hat{a}, \hat{a}] = [\hat{a}^\dagger, \hat{a}^\dagger] = 0$, $[\hat{a}, \hat{a}^\dagger] = 1$.
- (b) $\hat{H} = \omega \left(\hat{a}^\dagger \hat{a} + \frac{1}{2} \right) \left(= \frac{1}{2} \omega \left(\hat{a}^\dagger \hat{a} + \hat{a} \hat{a}^\dagger \right) \right)$.
- (c) $\hat{N} \equiv \hat{a}^\dagger \hat{a}$ is Hermitian (and thus has real eigenvalues).
- (d) Define $\hat{N}|n\rangle \equiv n|n\rangle$ and $\langle n|n\rangle \equiv 1 \Rightarrow \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$, $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$.

Exercise 2.2: In lecture we have determined $\hat{x}_H(t)$ starting from the e.o.m.. We will now determine it more directly

- (a) Show that

$$e^{\hat{A}} \hat{B} e^{-\hat{A}} = \hat{B} + [\hat{A}, \hat{B}] + \frac{1}{2!} [\hat{A}, [\hat{A}, \hat{B}]] + \frac{1}{3!} [\hat{A}, [\hat{A}, [\hat{A}, \hat{B}]]] + \dots$$

Hint: use the Cauchy product formula.

- (b) Determine the following operator with the help of the series expansion in a)

$$\hat{x}_H(t) \equiv e^{i\hat{H}t} \hat{x} e^{-i\hat{H}t}.$$

Exercise 2.3: Let us consider the Fock space of a free scalar field and define

$$\begin{aligned} |\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\rangle &\equiv \hat{a}_{\vec{k}_1}^\dagger \hat{a}_{\vec{k}_2}^\dagger \dots \hat{a}_{\vec{k}_n}^\dagger |0\rangle, \\ \hat{N} &\equiv \int d^3\vec{p} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}}, \\ :\hat{H}: &\equiv \int d^3\vec{p} E_{\vec{p}} \hat{a}_{\vec{p}}^\dagger \hat{a}_{\vec{p}}. \end{aligned}$$

What is then $\hat{N}|\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\rangle$ and $:\hat{H}:|\vec{k}_1, \vec{k}_2, \dots, \vec{k}_n\rangle$?**Exercise 2.4:**

- (a) Demonstrate that $|\vec{k}_1, \vec{k}_2\rangle = |\vec{k}_2, \vec{k}_1\rangle$. [In other words these are Bosons.]
- (b) Compute $\langle \vec{k}'_1, \vec{k}'_2 | \vec{k}_1, \vec{k}_2 \rangle$.