5.6. Effective action and background field method

- reminder generating functionals for scalar field theory (φ^4_{4}, φ^3_6, etc)

$$Z[J] = e^{W[J]} = \int \mathcal{D}\phi \, e^{-S_E[\phi] + \int d^4x \, \phi(x) J(x)}$$

<u>Legendre trafo</u> $\Gamma[\varphi] = W[J] - \int d^4x \, \varphi(x) J(x)$

$$\Rightarrow \varphi(x) = \frac{\delta W[J]}{\delta J(x)} \, , \quad J(x) = -\frac{\delta \Gamma[\varphi]}{\delta \varphi(x)}$$

- fields $\varphi(x)$, $J(x)$ are indep. of field $\phi(x)$
- all objects are commuting (not operator valued) as we use path int.

$$\Rightarrow e^{\Gamma[\varphi]} = e^{W[J]} \, e^{-\int dx^4 \varphi(x) J(x)}$$

$$= \int \mathcal{D}\phi \, e^{-S_E[\phi] + \int d^4x (\phi(x) - \varphi(x)) J(x)}$$

$$= \int \mathcal{D}\phi \, e^{-S_E[\phi] - \int d^4x (\phi(x) - \varphi(x)) \frac{\delta \Gamma[\varphi]}{\delta \varphi(x)}}$$

- $\Gamma[\varphi]$ was def as the generating functional for all 1PI Green's funct. it is also called <u>effective action</u>.

<u>saddle point approx.</u>

substitute $\phi(x) = \varphi(x) + \phi'(x)$

We now require the following boundary conditions:

$$\tilde{\phi}'(p=0) \equiv 0 \iff \int d^4x \, e^{-i(p=0)\cdot x} \phi'(x) = \int d^4x \, \phi'(x) = 0$$

id est we only keep nonvanishing momenta in $\tilde{\phi}'$ (subtracted const. mode)

If $\varphi(x)$ is a constant field (x-indep) we have

$$e^{\Gamma[\varphi]} = \int \mathcal{D}\phi' \, e^{-S_E[\varphi + \phi'] - \underbrace{\int d^4x \, \phi'(x)}_{0} \cdot \frac{\delta \Gamma}{\delta \varphi}} \qquad \Big| \text{drop terms linear in } \phi'$$

$$e^{\Gamma[\varphi]} = \int \mathcal{D}\phi' e^{-S_E[\varphi+\phi']} \Big|_{\text{no linear } \phi' \text{ terms}}$$

corresponds formally to a saddle point approximation (in some cases this may be generalised to $\varphi = \varphi(x)$).

· back to gauge fields

Here the source terms $\int d x^4 A_\mu^a(x) J_\mu^a(x)$ in general break gauge invariance. That is $\Gamma[A_\mu^a]$ is in general neither gauge invariant, nor indep of the gauge-fixing parameter ξ.

For $0 = J_\mu^a(x) = \dfrac{\delta\Gamma[A_\mu^a]}{\delta A_\mu^a(x)} = 0$ we may recover gauge inv (apart from gauge fixing). This means that physical observables are only consistently defined when using this "on-shell" condition

((we already exploited the inv. for phys. Green's funct to derive the Ward-Takahashi identities))

Background gauge

· The determination of $\Gamma[A_\mu^a]$ can be simplified in this method. Because the final answer for Γ should be gauge indep. we may use a particular choice $\Rightarrow$ symmetry on Γ, easier

The background field A_μ^a in background gauge is called B_μ^a

((next week: use results from this for asymptotic freedom)) consider a $SU(N)$ valued field $\Phi = \Phi^a T^a$

- $D_\mu(A) = \partial_\mu - ig A_\mu$ is the cov. derivative

$$\Rightarrow [D_\mu(A), \Phi] = \partial_\mu \Phi^a T^a - ig [A_\mu^a T^a, \Phi^b T^b]$$
$$= \partial_\mu \Phi^a T^a + g f^{abc} A_\mu^a \Phi^b T^c \qquad f^{abc} = f^{cab}$$
$$= T^a \underbrace{(\delta^{ab} \partial_\mu + g f^{abd} A_\mu^d)}_{\equiv \mathcal{D}_\mu^{ab}(A)} \Phi^b$$

where we call $\mathcal{D}_\mu^{ab}(A)$ the covariant derivative in the adjoint representation. For more about this see ex. 12.4.

we will shift $A_\mu^a \to A_\mu^a + B_\mu^a$ by the background field (corresp. to $\theta(x)$, with b.c.)

a) <u>before the shift</u> we have $S_E = S_E[A_\mu^a]$ as usual

* for the gauge fixing term we choose

$$G^a \equiv -\mathcal{D}_\mu^{ab}(B)(A_\mu^b - B_\mu)$$

$\Rightarrow$ gauge trafo on A (p. 87) by infinitesimal Θ^a

$$A_\mu' = A_\mu + \partial_\mu \Theta^a(x) - g f^{bca} \Theta^b A_\mu^c + \mathcal{O}(\Theta^2)$$

$$\Rightarrow \frac{\partial A_\mu^a}{\partial \Theta^b} = \mathcal{D}_\mu^{ab}(A) \quad \text{using the above}$$

$$\Rightarrow \frac{\partial G^a}{\partial \Theta^b} = -\mathcal{D}_\mu^{ac}(B)\, \mathcal{D}_\mu^{cb}(A)$$

$\Rightarrow$ b) <u>after the shift</u> we have

$$S_E[A_\mu^a + B_\mu^a], \quad \boxed{G^a = -\mathcal{D}_\mu^{ab}(B) A_\mu^b}$$

$$\frac{\partial G^a}{\partial \Theta^b} = -\mathcal{D}_\mu^{ac}(B)\, \mathcal{D}_\mu^{cb}(A+B)$$

Upon the appropriate choice of B_μ^a we have as before as a SP approx

$$e^{\Gamma[B]} = \int \mathcal{D}A_\mu^a \mathcal{D}\bar{c}^a \mathcal{D}c^a \exp\left[-S_E[A_\mu^a + B_\mu] - \int d^4x \frac{1}{2\xi} G^a G^a + \bar{c}^a \frac{\partial G^a}{\partial \Theta^b} c^b\right]$$

- we now perform a gauge trafo on the background field B_μ^a (and not on A_μ^a)

$$B_\mu^a \to B_\mu'^a = U B_\mu^a U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger$$

- the integration measures $\mathcal{D}A$, $\mathcal{D}\bar{c}$, $\mathcal{D}c$ are invariant under the unitary trafos

$$A_\mu \to U A_\mu U^\dagger$$
$$\bar{c} \to U \bar{c} U^\dagger$$
$$c \to U c U^\dagger$$

$$\Rightarrow \quad A_\mu + B_\mu \to U(A_\mu + B_\mu)U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger$$

like a gauge trafo of

- $\Gamma[B]$ is invariant under this gauge trafo of B as

i) $S_E[A_\mu^a + B_\mu^a]$ is gauge inv

ii) $-T^a G'^a = T^a \mathcal{D}_\mu^{ab}(B') A_\mu'^b \overset{\Phi = A_\mu}{=} [D_\mu(B'), A_\mu']$

$-G'$ p. 80 $= [U D_\mu(B) U^\dagger, U A_\mu U^\dagger] = U[D_\mu(B), A_\mu]U^\dagger = -UGU^\dagger$

$$\Rightarrow \quad \frac{1}{2} G'^a G'^a = \mathrm{Tr}[T^a G'^a T^b G'^b] = \mathrm{Tr}[G' G'] = \mathrm{Tr}[GG]$$
$$= \frac{1}{2} G^a G^a \quad \Rightarrow \text{the GF term is inv.!}$$

iii) ghost term:

ex 12.1: $\int d^4x\, \bar{c}^a \frac{\partial G^a}{\partial \Theta^b} c^b = \int d^4x\, 2\,\mathrm{Tr}\{[D_\mu(B), \bar{c}][D_\mu(A+B), c]\}$

which is invariant

Does this trafo introduce a Jacobian?

(no sum convention): $\sum_a A_\mu'^a A_\mu'^a = 2\sum_{a,b} \mathrm{Tr}[T^a A_\mu'^a T^b A_\mu'^b] = 2\,\mathrm{Tr}[A_\mu' A_\mu']$

$$= 2\,\mathrm{Tr}[U A_\mu U^\dagger U A_\mu U^\dagger] = \sum_a A_\mu^a A_\mu^a$$

→ the scalar product is preserved,

orthogonal trafo with $\det\left[\frac{\partial A_\mu'^c}{\partial A_\mu^d}\right] = 1$

(ditto for $\mathcal{D}\bar{c}, \mathcal{D}c$)

<u>Consequences</u> for renormalisation $g_B \equiv Z_g^{\frac{1}{2}} g_R$, $B_B = Z_B^{\frac{1}{2}} B_R$

We saw for ϕ^4-theory how to derive the RG eq for the renormalised λ_R from the renormalisation factor Z_λ (see p. 62.)

- the background field method simplifies the determination of Z_g here:

<u>observation</u>: $\Gamma[B] = \sum_n \frac{1}{n!} \underbrace{\Gamma_B^{(n)}}_{Z_B^{-\frac{n}{2}} \Gamma_R} \underbrace{B_B^n}_{Z_B^{\frac{n}{2}} B_R^n}$, p. 58 the fields and $\Gamma^{(n)} \sim A^{(n)}$ renormalise in the opposite way, $\Gamma[B]$ is <u>finite</u> express. through R quantities!

to LO. $\Gamma[B] = \overset{\text{const}}{C'} \int d^4x\, \mathcal{L}_{eff}[B]$

symbolically (no indices) $= C' \int [\partial B_B \partial B_B + g_B \partial B_B\, B_B^2 + g_B^2 B_B^4] + \dots$

$$= C' \int [Z_B (\partial B)^2 + Z_B^{\frac{3}{2}} Z_g^{\frac{1}{2}} g_R \partial B_R \cdot B_R^2 + Z_B^2 Z_g g_R^2 B_R^4] + \dots$$

is finite $\Rightarrow$ $C' Z_B$ = finite

$\Rightarrow$ $Z_B Z_g$ = finite, so Z_B determines Z_g

☆ the simplification here lies in the fact that Z_B follows from the <u>quadratic</u> part of the action, and usually we need 3- and 4-vertices to compute Z_g !