Topics in Random Matrix Theory   Summer Term 2017

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- Lecture: Mondays 10.15 - 11.45 05-153
- Exercises — " — 14.15 - 15.00 — " —

no lectures on: 1.5., 5.6., 26.6.

- Literature: - standard books see in arXiv 1603.06011: references [13 - 16] and review [12]
  - more as we go

* successful participation (apart from attending lectures):
  - active participation in Exercises (will be given in Lecture)
  - Oral exam (→ credit points)

* goal: give an overview over some modern developments, often without detailed derivation (which can be found in the literature)

Part I: Non-Hermitian One- and multi-matrix models see arXiv

Chapter 1: Non-Hermitian Random Matrix Theory (RMT)

Examples: consider an $N \times N$ matrix $J \in \mathbb{C}^{N\times N}, \mathbb{R}^{N\times N}$ $(\mathbb{H}^{N\times N})$

with $J^{\dagger} = \bar{J}^{T} \neq J$ i.e. $\begin{cases}\text{complex non-Hermitian} \\ \text{real asymmetric}\end{cases}$

with distribution of matrix elements

$$\boxed{P(J) = c \; \exp[-\mathrm{Tr}\, J J^{\dagger}]} = c \exp[-\sum_{i,j=1}^{N} \underbrace{J_{ij} J^{\dagger}_{ji}}_{|J_{ij}|^2}]$$

product of Gauß' distrib $\forall$ indep $\left\{ \begin{matrix} (2N)^2 \\ N^2 \end{matrix} \right\}$ matrix elements J_{ij}

$\Rightarrow C = (\sqrt{\pi})^{-(2N)^2}$ for $J_{ij} \in \mathbb{C}$ (often we don't normalise $Z_N = \int [dJ]\, P(J)$)
$[dJ]$ flat Lebesgue measure over all indep $\mathrm{Re}\, J_{ij}, \mathrm{Im}\, J_{ij}$

This is called the <u>complex</u> or <u>real Ginibre ensemble</u> [J. Ginibre, J. Math. Phys 6 (1965) 440]

Q: What is the distribution of eigenvalues (ev) $z_{j=1,\dots,N}$ of J

$J\psi = z\psi \iff \underline{0 = \det(z\mathbb{1}_N - J)}$ or more general

What are correlations amongst ev?

- $J_{ij} \in \mathbb{R} \Rightarrow$ char. eq is real, thus z_j real or in complex conj. pairs
  very difficult problem, first solution 2007!
- $J_{ij} \in \mathbb{C}$ much easier, was solved by Ginibre,
  we focus on this:

Strategy: – find joint density of complex ev $P(J) \to P(z_1, \dots, z_N)$
by <u>Schur decomp</u> $\boxed{J = U(Z + T)U^{\dagger}}$, $U \in U(N)/U(1)^N$
$Z = \mathrm{diag}(z_1 \dots z_N)$, $T_{ij} \in \mathbb{C}$ complex upper triangular,
by computing Jacobian & integrating out U and T
$\Rightarrow$ determinantal point process

- solve with <u>orthogonal Polynomials (OP) on $\mathbb{C}$</u>, major tool in this lecture series

<u>Motivation</u>: Why non-Hermitian matrices with complex ev.?

- <u>adjacency matrix</u> A of a graph with N nodes:
  $A_{ij} = 1$ iff $\exists$ link from node i to node j $\quad i \longrightarrow j$, $A_{ij} = 0$ else
  is a <u>directed graph</u> if $A_{ij} = 1 \not\Rightarrow A_{ji} = 1$
  if $\{0,1\}$ are random variables $A \neq A^T$ is such a random matrix with real + complex eigenvalue pairs

- <u>QCD - Dirac Operator $\not{D}$</u> with chemical potential; see 1603.06011
  in certain limits Hamiltonians or $\not{D}$ can be approx by RMT

$$\not{D} = \begin{pmatrix} 0 & iW \\ iW^+ & 0 \end{pmatrix} \quad \text{anti-Hermitian} \quad \not{D}^+ = -\not{D}.$$

from $0 = \{\not{D}, \gamma_5\} = \not{D}\gamma_5 + \gamma_5\not{D}$ in 4 Euclidean dim, $\gamma_5 = \begin{pmatrix} 1\!\!1_2 & 0 \\ 0 & -1\!\!1_2 \end{pmatrix}$
($\hookrightarrow$ Clifford algebra)

with Lagrangian $\mathcal{L}_{\psi} = \bar{\psi}(\not{D} + m)\psi +$ $\quad$ ψ Dirac Fermion
+ chemical potential $\mu \in \mathbb{R}$: add $\mu\psi^+\psi = \bar{\psi}\gamma_0\mu\psi$ $\quad$ $(\bar{\psi} = \psi^+\gamma_0,\ \gamma_0 = \begin{pmatrix} 0 & 1\!\!1_2 \\ 1\!\!1_2 & 0 \end{pmatrix})$

$$\Rightarrow \text{consider } "\not{D} + \mu" = \begin{pmatrix} 0 & \underbrace{iJ_1 + \mu J_2}_{C} \\ \underbrace{iJ_1^+ + \mu J_2^+}_{D} & 0 \end{pmatrix} = \not{D}_{RMT} \neq \not{D}_{RMT}^{+};\ J_{1,2} \text{ Ginibre}$$

$$\Rightarrow \text{ev } z_j \in \mathbb{C}, \text{ and } \det(z - \not{D}_{RMT}) = \underline{\det(z^2 - CD)} = 0$$

we need the ev of the <u>product</u> of 2 coupled random matrices

<u>Advanced questions</u>:

- what happens for other distributions of matrix elements, are the <u>eigenvalue correlations universal</u> for $N \to \infty$, that is are they independent of details of the distrib. $P(J)$? Examples are e.g. Wigner matrices : J_{ij} iid random variables, not all Gaussian with equal variances

e.g. non-Gaussian invariant measures $P(J) \sim \exp[-\mathrm{Tr}\, V(J, J^\dagger)]$

the difficulty is to decouple Z from U and T (unlike for $J = J^\dagger$ for Hermitian RMT)

consider $\mathrm{Tr}(JJ^\dagger) = \mathrm{Tr}[(U(Z+T)U^\dagger)(U(Z^\dagger + T^\dagger)U^\dagger)]$

$$= \mathrm{Tr}\, ZZ^\dagger + \underbrace{\mathrm{Tr}(ZT^\dagger + TZ^\dagger)}_{0} + \mathrm{Tr}\, TT^\dagger$$ decouple in exp

Ex 1 show that $\mathrm{Tr}[(JJ^\dagger)^2]$ does not decouple

$$\mathrm{Tr}\, J^2 = \mathrm{Tr}\, Z^2 \ , \ \mathrm{Tr}\, J^{\dagger 2} = \mathrm{Tr}(Z^\dagger)^2$$

$\Rightarrow$ in $V(J, J^\dagger) = a JJ^\dagger + v(J) + v(J^\dagger)$, with $v(J)$ polynomial Z and U, T decouple. These are called harmonic potentials.

However:

Ex 2: show that for $v(J)$ of degree ≥ 3 the integral $\int_{\mathbb{C}} dz\, e^{-V(z,\bar{z})}$ will not converge, by giving an example

$\Rightarrow$ The elliptic Ginibre ensemble $\boxed{P_\tau(J) = c \exp[-a \mathrm{Tr}\, JJ^\dagger + b\, \mathrm{Tr}(J^2 + J^{\dagger 2})]}$

with $a = \frac{1}{1-\tau^2}$, $b = \frac{\tau}{2(1-\tau^2)}$, $\tau \in [0,1)$ is an example for a general ensemble that is quadratic in $J, J^\dagger$ and convergent. It extrapolates between the Ginibre ($\tau = 0$) and the Gaussian Unitary Ensemble ($\tau = 1$)

Ex 3 show this on the level of distribution of matrix elements

* For normal matrices, $[J, J^\dagger] = 0 \Rightarrow J = UZU^\dagger$ more general invariant measures are possible

[Note: For the same distrib. of matrix el. $P(J) = c\, e^{-\mathrm{Tr}\, JJ^\dagger}$ we could ask different Q's, distrib. & correl. of singular values of $\boxed{J = U \Lambda V}$ $U, V \in U(N)$, $\Lambda = \mathrm{diag}(\lambda_1 \ldots \lambda_N)$, $\lambda_j \geq 0$, $\Leftrightarrow$ ev of $JJ^\dagger$ pos. Hermitian matrix. This is considered a "different ensemble of random matrices": Wishart-Laguerre Unitary or chiral Gaussian Unitary Ensemble]

(DPP)

Determinantal point processes in the complex plane

Def $$\boxed{P_N(z_1,\dots,z_N) = \prod_{j=1}^{N} \omega(z_j)\ |\Delta_N(z)|^2}$$ (un normalised) joint density of ev (jpdf)

where $\Delta_N(z) = \prod_{j>i}^{N} (z_j - z_i) = \begin{vmatrix} 1 & z_1 & \cdots & z_1^{N-1} \\ \vdots & & & \\ 1 & z_N & & z_N^{N-1} \end{vmatrix}$ Vandermonde determinant

and $Z_N = \prod_{i=1}^{N} \int_{\mathbb{C}} d^2 z_i\, \omega(z_i)\ |\Delta_N(z)|^2$ partition function

Examples:
- elliptic Ginibre ensemble $\boxed{\omega(z) = \exp[-\frac{1}{1-\tau^2}(|z|^2 - \frac{\tau}{2}(z^2+\bar{z}^2))]}$
  see arXiv chao-dyn/9802025
- rectangular Ginibre ensemble $\boxed{\omega_\nu(z) = |z|^{2\nu} e^{-|z|^2},\ \nu > -1}$
  see arXiv 1107.5019
  for $\nu \in \mathbb{N}$ from $J: N \times (N+\nu)$ complex matrix

Notice: for real Ginibre matrices the jpdf is not $\sim |\Delta_N(z)|^1$
for quaternion " " " " " " " $\sim |\Delta_N(z)|^4$

* we still need to show that P_N is a DDP $\to$

Method of orthogonal Polynomials in the complex plane

the polynomials $P_n(z) = z^n + \dots$ that we take in monic normal.
are called orthogonal wrt weight $\omega(z)$ if

$$\langle P_n, P_\ell \rangle \equiv \boxed{\int_{\mathbb{C}} d^2 z\, \omega(z)\, P_n(z)\, \overline{P_\ell(z)} = h_n\, \delta_{n,\ell}}\ ,\quad h_n = \|P_n\|^2 \text{ squared norms},$$

In the following we will mainly consider positive weight functions $\omega(z) > 0\ \forall z \in \mathbb{C}$, defining a proper scalar product $\langle\, ,\, \rangle$

<u>Note</u> however, that physically relevant situations exist, where $\omega(z) \in \mathbb{C}$ and $\langle , \rangle$ is not a positive definite inner product, e.g. for QCD + μ and weight $\prod_{f=1}^{N_f} \det(D_{\mu\nu} + m_f)\, e^{-\mathrm{Tr}\, J_1 J_1^\dagger - \mathrm{Tr}\, J_2 J_2^\dagger}$ for N_f Fermion flavours. In general one can still def. OP $P_n(z)$ and $\overline{Q_\ell(z)}$ $(\neq \overline{P_\ell(z)})$ s.th. $\int_{\mathbb{C}} d^2z\, \omega(z)\, P_n(z)\, \overline{Q_\ell(z)} = \delta_{n\ell} h_\ell$

<u>Gram-Schmidt construction</u> (holds for above setup too!)

define moments $\mu_{ij} = \int d^2z\, \omega(z)\, z^i \bar{z}^j = \langle i, j \rangle \neq \langle j, i \rangle$

$$\Rightarrow \quad P_n(z) = \frac{\begin{vmatrix} \langle 0,0 \rangle & \cdots & \langle n-1, 0 \rangle & \langle n, 0 \rangle \\ \vdots & & \vdots & \\ \langle 0, n-1 \rangle & \cdots & \langle n-1, n-1 \rangle & \langle n, n-1 \rangle \\ 1 & & z^{n-1} & z^n \end{vmatrix}}{\Delta_{n-1}} , \quad \Delta_n = \begin{vmatrix} \langle 0,0 \rangle & \cdots & \langle n, 0 \rangle \\ \langle 0, n \rangle & \cdots & \langle n, n \rangle \end{vmatrix}$$

is a <u>monic OP</u> wrt $\omega(z)$: <u>Ex 4</u> show this (and construct Q_ℓ for ω in general)

as $\int d^2z\, \omega(z)\, P_n(z)\, \bar{z}^j = 0 \quad \forall j = 0, 1, \ldots, n-1$, $\quad h_n = \frac{\Delta_n}{\Delta_{n-1}}$

$\Rightarrow$ we can define the <u>kernel of OP</u>

$$\boxed{K_N(z, \bar{u}) = \sum_{j=0}^{N-1} \frac{1}{h_j} P_n(z)\, \overline{P_n(u)}}$$

and it hold $P_N(z_1, \ldots, z_N) = \prod_{j=1}^{N} \left(h_{j-1}\, \omega(z_j) \right) \det_{1 \le i,j \le N} \left[K_N(z_i, \bar{z}_j) \right]$ is a DPP

with $Z_N = N! \, \prod_{j=0}^{N-1} h_j$ (proove this!)

$\perp$ wavefunction $\psi_{j-1}(z_k)$

$\Big[$ the proof is the same as on $\mathbb{R}$: $\Delta_N(z) = \det_{1 \le j,k \le N} \left[P_{j-1}(z_k) \right] = \prod_{\ell=0}^{N-1} h_\ell^{\frac{1}{2}} \det_{j,k} \Big[\underbrace{\frac{P_{j-1}(z_k)}{h_{j-1}^{\frac{1}{2}}}}_{A_{kj}} \Big]$

and $|\Delta_N(z)|^2 \underset{\det A = \det A^T}{=} \det_{ij} [A_{ij}] \det_{ij} [\bar{A}_{ji}] \prod_{j=0}^{N-1} h_j = \det_{i,k} \Big[\sum_{j=1}^{N} A_{ij} \bar{A}_{jk} \Big] \prod_{\ell=0}^{N-1} h_\ell = \det_{i,k} [K_N(z_i, \bar{z}_k)] \prod_{\ell=0}^{N-1} h_\ell$ $\Big]$

The kernel satisfies the conditions for Dysons Theorem: (proof known?)
(Thm 5.1.4, Mehta, Random Matrices, 3rd Edition, Elsevier, NY 2004)

that $\overline{K_N(z,\bar{u})} = K_N(u,\bar{z})$, $\int_{\mathbb{C}} d^2z\, \omega(z) K_N(z,\bar{z}) = N$

and $\int_{\mathbb{C}} d^2u\, \omega(u)\, K_N(z,\bar{u})\, K_N(u,\bar{v}) = K_N(z,\bar{v})$ which is easy to show

((in Mehta $\kappa_N(z,\bar{u}) = \omega(z)^{\frac{1}{2}}\omega(u)^{\frac{1}{2}} K_N(z,\bar{u})$))

$$\Rightarrow \boxed{\int_{\mathbb{C}} d^2z_L\, \omega(z_L) \det_{1\le i,j\le L}\left[K_N(z_i,\bar{z}_j)\right] = (N-L+1) \det_{1\le i,j\le L-1}\left[K_N(z_i,\bar{z}_j)\right]}$$

and hence we can express all k-point-correlation functions

$$\underline{R_k(z_1,\dots,z_k)} \equiv \frac{N!}{(N-k)!}\frac{1}{Z_N}\int d^2z_{k+1}\dots d^2z_N\, P_N(z_1,\dots,z_N)$$
$$= \prod_{j=1}^{k}\omega(z_j) \det_{1\le l,m\le k}\left[K_N(z_l,\bar{z}_m)\right]$$

representations of $P_n(z)$ and $K_N(z,\bar{u})$ as expectation values of characteristic polynomials

$$\boxed{P_n(z) = \left\langle \prod_{j=1}^{n}(z-z_j)\right\rangle_n} = \frac{1}{Z_n}\int d^2z_1\dots d^2z_n\, \omega(z_1)\dots\omega(z_n) \prod_{j=1}^{n}(z-z_j)\, P_n(z_1\dots z_n)\rangle$$

and $\left(= \langle \det(z\mathbb{1}_n - J)\rangle_n \text{ for } J\; n\times n \text{ matrix in case } \exists \text{ such a rep. of } P_n(z_1,\dots,z_n)\right)$
$\qquad = \mathbb{E}(z\mathbb{1}_n - J)$

$$\boxed{K_N(z,\bar{u}) = \frac{1}{h_{N-1}}\left\langle \prod_{j=1}^{N-1}(z-z_j)\prod_{j=1}^{N-1}(\bar{u}-\bar{z}_j)\right\rangle_{N-1}} \left(= \frac{1}{h_{N-1}}\langle \det(z\mathbb{1}_{N-1} - J)(\bar{z}\mathbb{1}_{N-1} - J^\dagger)\rangle_{N-1}\right)$$

$\underline{Ex\ 5}$: proof of these 2 statements expanding $\det P_{j-1}(z_k)$ (as on $\mathbb{R}$)

Note that in general on $\mathbb{C}$ neither do $P_k(z)$ satisfy a 3-step recurrence relation, nor $K_N(z,\bar{u})$ a Christoffel-Darboux identity!