Summary Lecture 10

The large-N limits of the Ginibre ensemble:

from the kernel weight $\mathcal{K}_N(z,\bar u) = e^{-\frac{N}{2}(|z|^2+|u|^2)} \sum_{l=0}^{N-1} \frac{(N z\bar u)^l}{\pi\, l!}$

we have obtained the following 3 limiting densities:

global density
= the circular law

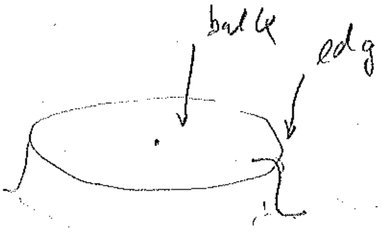
$$\boxed{\rho(z) = \lim_{N\to\infty} \frac{\Gamma(N, N|z|^2)}{\pi\,\Gamma(N)} = \frac{1}{\pi}\Theta(1-|z|)}$$

local density/kernel in the bulk (we showed this at the origin)

$$\lim_{N\to\infty} K_N\left(\frac{z}{\sqrt N}, \frac{\bar u}{\sqrt N}\right) = e^{-\frac{|z|^2}{2}-\frac{|u|^2}{2}}\, e^{z\bar u} \quad , \text{ e.g. } R_2 = \frac{1}{\pi^2}\left(1-e^{-|z-u|^2}\right)$$

local density at the edge

$$\lim_{N\to\infty} \frac{\Gamma(N, |\sqrt N + x|^2)}{\pi\,\Gamma(N)} = \frac{1}{2\pi}\,\mathrm{erfc}(\sqrt{2}\,x) = \rho_{edge}(x)$$



The large-N limit of the elliptic Ginibre ensemble

global: circular $\to$ elliptic law

local bulk kernel remains the same as for Ginibre, universal
(" edge also)

$\exists$ one further limit: weak non-Hermiticity Ginibre $\leftrightarrow$ GUE

$$\underset{\downarrow\,\infty}{N}\,(\underbrace{1-\tau^2}_{\downarrow\,0}) = \alpha^2 \quad \text{constant}$$

Universality of the elliptic Ginibre ensemble:

"naive" limit
$$\lim_{N\to\infty} K_N(z,\bar{u}) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\left(\frac{\tau}{2}\right)^\ell He_\ell\left(\frac{z}{\sqrt{2\tau}}\right) He_\ell\left(\frac{\bar{u}}{\sqrt{2\tau}}\right)$$

Mehler formula
$$= \frac{1}{\sqrt{1-\tau^2}} \exp\left[\frac{1}{1-\tau^2}\left(z\bar{u} - \frac{\tau}{2}\left(z^2+\bar{u}^2\right)\right)\right]$$

Exercise 22, check in the limit $\tau\to 0$

= k-point correl. function

$$R_k(z_1,\dots,z_k) \sim \prod_{j=1}^{k} e^{-\frac{1}{1-\tau^2}\left(|z_j|^2 - \frac{\tau}{2}(z_j^2+\bar{z}_j^2)\right)} \det_{1\le \ell,m\le k}\left[\frac{1}{\sqrt{1-\tau^2}}\, e^{\frac{z_\ell \bar{z}_m}{1-\tau^2} - \frac{\frac{\tau}{2}\left(z_\ell^2+\bar{z}_m^2\right)}{1-\tau^2}}\right]$$

take out and cancel

$$= \frac{1}{(1-\tau^2)^{\frac{k}{2}}} \prod_{j=1}^{k} e^{-\frac{1}{1-\tau^2}|z_j|^2} \det_{1\le \ell,m\le k}\left[e^{\frac{z_\ell \bar{z}_m}{1-\tau^2}}\right]$$

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same as for Ginibre kernel, after rescaling $z_\ell \to \frac{z_\ell}{\sqrt{1-\tau^2}}$

= universal!

Weak non-Hermiticity

* we consider simultaneously (= double scaling limit)
  $N\to\infty$ and $\tau\to 1$, the Hermitian limit: $\lim_{\tau\to 1} \frac{1}{\sqrt{\pi(1-\tau)}} e^{-\frac{y^2}{1-\tau}} = \delta(y)$
  such that $\boxed{\lim_{\substack{N\to\infty\\ \tau\to 1}} N(1-\tau^2) = \alpha^2}$ is kept constant

$\Rightarrow$ the global density (Ellipse) collapses to the GUE semicircle
But: the local correlations still extend into the complex plane $\mathbb{C}$

* scaling of z_j's:
  - for compact support we need $z_j \to \sqrt{N} z_j$, weight $e^{-\frac{N}{1-\tau^2}\left(|z|^2 - \frac{\tau}{2}(z^2+\bar{z}^2)\right)}$

then $z_{1,2} = X + \frac{x_{1,2}}{N} + i\frac{y_{1,2}}{N} = X + \frac{\xi_{1,2}}{N}$ ($X\in\mathbb{R}$, $\xi_{1,2}\in\mathbb{C}$, $\frac{1}{N}$ deviations from $\mathbb{R}$

weight function: $\omega(z_1) = \exp\left[-\frac{N}{1+\tau}\left(X+\frac{x_1}{N}\right)^2 - \frac{N}{1-\tau}\left(\frac{y_1}{N}\right)^2\right]$

$$\xrightarrow[\substack{N\to\infty\\ \tau\to 1}]{} \exp\left[-\frac{N}{2}X^2 - X x_1 - 2\frac{y_1^2}{\alpha^2}\right]$$

↑ GUE weight

i.e. ∃ remaining non-trivial contrib.

* for the kernel the limit is more tricky, involving $\int$-rep's of the Hermite polyn. and a carefull saddle point analysis (see Lit). We cannot use the Mehler formula here, (non uniform)!

result: $\lim\limits_{\substack{N\to\infty\\ \tau\to 1}} \frac{\omega(z_1)^{\frac{1}{2}}\omega(\bar{z}_2)^{\frac{1}{2}} K_N(z_1,\bar{z}_2)}{N^2} \sim \frac{e^{-\frac{y_1^2}{\alpha^2}-\frac{y_2^2}{\alpha^2}}}{\alpha}\int_0^1 du\, e^{-\frac{\alpha^2 u^2}{2}}\cos(u(\xi_1-\bar{\xi}_2))$

(here for $X=0$ at the origin, else $\int_0^1 \to \int_0^{\sqrt{1-X^2/2}}$)

↑ this is the kernel $K(\xi_1,\bar{\xi}_2;\alpha)$ (in form of hep-th/0206086)

Interpolating between Sine-kernel & Ginibre:

GUE limit $\alpha\to 0$: $\lim\limits_{\alpha\to 0}\frac{1}{\sqrt{\pi}\alpha}e^{-\frac{y^2}{\alpha}} = \delta(y)$ collapse to $\mathbb{R}$

and $\int_0^1 du\, 1\cdot\cos(u(x_1-x_2)) = \frac{1}{x_1-x_2}\left[\sin(u(x_1-x_2))\right]_0^1 = \frac{\sin(x_1-x_2)}{x_1-x_2}$

Ginibre limit $\alpha\to\infty$: here the scaling of z_i is differnt ($\sqrt{N}$ vs N in weak lim)

so we have to rescale ξ_i's with $\alpha\sim\sqrt{N}$

and one can show that $\lim\limits_{\alpha\to\infty} K(\xi_1,\bar{\xi}_2;\alpha)\Big|_{\frac{\xi_{1,2}}{\alpha}\text{ fixed}} \sim K_{Gin}$

Exercise 28 sketch this limit

- The large-N limit for products of $\underline{m}$ indep Ginibre matrices

* What changes: – the scaling in N as function of m
  – the global density ≠ circular law
  – local bulk and edge scaling remain = universal
  – there is a new local origin scaling limit

[for local limits 1208.0187]

weights. Kernel: (p. 28)

$$w_m(z)^{\frac{1}{2}} w_m(u)^{\frac{1}{2}} K_N(z,\bar{u}) = G^{m0}_{0m}(\vec{0}\,|\,|z|^2)^{\frac{1}{2}}\, G^{m0}_{0m}(\vec{0}\,|\,|u|^2)^{\frac{1}{2}} \sum_{\ell=0}^{N-1} \frac{(z\bar{u})^\ell}{\pi(\ell!)^m}$$

density

$$R_1(z) = G^{m0}_{0m}(\vec{0}\,|\,|z|^2) \sum_{\ell=0}^{N-1} \frac{|z|^{2\ell}}{\pi(\ell!)^2}$$

* $m=1$: back to Ginibre, there we had to rescale $z \to N^{\frac{1}{2}} z$ to get compact support and the circular law

here Ansatz $\boxed{z \to N^{\delta} z}$, we will find $\delta = \frac{m}{2}$

$\Rightarrow$ we need the large-argument $N^{2\delta}|z|^2$ asymptotic of G^{m0}_{0m} and of the sum

* except at the local origin limit $z \to N^{\delta} z \to N^{\delta}\frac{z}{N^{\delta}}$ undoing this scaling

("naive" limit): $\lim_{N\to\infty} w_m(z) K_N(z,\bar{z}) = \boxed{\frac{1}{\pi} G^{m0}_{0m}(\vec{0}\,|\,|z|^2) \underbrace{\sum_{\ell=0}^{\infty} \frac{|z|^{2\ell}}{(\ell!)^m}}_{{}_0F_{m-1}(-;\vec{1};|z|^2)} = \rho^{(m)}_{\text{origin}}(z)}$

${}_0F_{m-1}(-;\vec{1};|z|^2)$ gen. hypergeometric function

example $m=2$ (p. 15)

$$\lim_{N\to\infty} R_1(z) = \frac{2}{\pi} K_0(2|z|)\, I_0(2|z|) \quad \neq \text{constant}$$

modified Bessel fct. of 2nd and 1st kind, Global & local bulk and edge densities:

• large-argument asymptotic of Meijer G-function C. J. L. Fields, Math. Comp 119, (1972), 7573

or use $G^{m0}_{0m}(\vec{0}\,|\,|z|^2) = 2^{m-1}\int_0^\infty \frac{dv_2}{v_2}\cdots\int_0^\infty \frac{dv_m}{v_m}\, e^{-\frac{|z|^2}{(v_2\cdots v_m)^2}-v_2^2-\cdots-v_m^2}$

Exercise 29 multiple saddle point analysis

$$\Rightarrow \lim_{|z|^2\to\infty} G^{m0}_{0m}(\vec{0}\,|\,|z|^2) \sim c\,|z|^{\frac{1-m}{m}}\, e^{-m|z|^{\frac{2}{m}}}$$ fractional powers

• asymptotic of hypergeometric kernel for large N and argument $|z| \to N^{\delta} z$

$$K_N(z,\bar z) = \sum_{k=0}^{N-1} \frac{|z|^{2k}}{\pi (k!)^m} \simeq (2\pi^3)^{-\frac{m}{2}} \int_1^N dk\, k^{-\frac{m}{2}}\, e^{k\ln|z|^2 - mk\ln k + mk}$$

using Stirling $k! \simeq \sqrt{2\pi k}\,\left(\frac{k}{e}\right)^k$ and $\sum \approx \int$

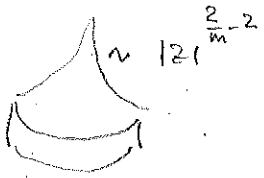
Exercise 30 use standard saddle point analysis to show

$$K_N(z,\bar z) \simeq c\,|z|^{\frac{1-m}{m}}\, e^{m|z|^{\frac{2}{m}}}\, \mathrm{erfc}\left(\frac{\sqrt{m}\,(|z|^{\frac{2}{m}}-N)}{\sqrt{2}\,|z|^{\frac{1}{m}}}\right)$$

* from these results, after appropriate rescaling $|z| \to |z|^{\frac{m}{2}}$ we obtain the same local bulk and edge limiting density as for Ginibre $\Rightarrow$ universality for more ensembles! (check)

* for the global density we obtain

$$\lim_{N\to\infty} N^{m-1} R_1(N^{\frac{m}{2}} z) = \boxed{\frac{|z|^{\frac{2}{m}-2}}{m\pi}\,\Theta(1-|z|) = \rho_m(z)}$$



$\neq$ circular law ("Halloween hat") at $m=1$

* the global density equals the density of a single Ginibre matrix to the power m: $(J)^m = U(Z+T)^m U^+$

$$\Rightarrow \mathrm{Tr}(JJ^+)^m = \sum_{j=1}^{N} z_j^m$$

change of variables: $s = r^m \Leftrightarrow s^{\frac{1}{m}} = r$

$$1 = \int_0^\infty dr\, r\, \rho(r) \cdot 2\pi = \int_0^\infty ds\, \frac{1}{m} s^{\frac{1}{m}-1} s^{\frac{1}{m}} \rho(s^{\frac{1}{m}})\, 2\pi = \int_0^\infty ds\, s\, \tilde{\rho}(s)\, 2\pi$$

e.g. density of single Ginibre — density of J^m

with $\tilde{\rho}(s) = \frac{1}{m} s^{\frac{2}{m}-2}$

- Product of m indep Ginibre $=$ (1 Ginibre)^m? for global density of complex ev yes, for local statistics no (for singular values not true for global density either

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Integrals over the unitary group $U(N)$

Literature
math-ph/0209030
hep-th/0002161

- In the determination of the joint density of eigenvalues (ev) so far the unitary matrices that diagonalise to an ev basis have dropped out $\Rightarrow \int_{U(N)\,\mathrm{Haar}} dU =$ const in the normalisation
- In more general 1- or 2- matrix models this is not always the case
  $\Rightarrow$ I'll present a list of 3 known integrals where the answer is known for arbitrary N

* for integrals over $O(N)$ or $USp(2N)$ this is not true in general!

$\rightarrow$ the technique to be applied for $U(N)$ is called character expansion