Summary   Lecture 11

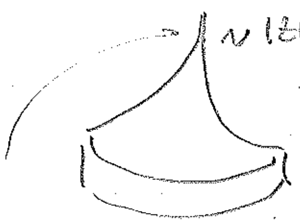
* we have encountered more Large-N limits in non-Hermitian RMT than possible in the Ginibre ensemble:

- weak non-Hermiticity limit: interpolates between sine-kernel in GUE and Ginibre-kernel in Ginibre

  $N(1-\tau^2)=\alpha^2$, $N \to \infty$, $\tau \to 1$

- products of m Ginibre matrices $Y_m = J_1 \cdots J_m$

  - global density $\sim |z|^{\frac{2}{m}-2}$ $\quad m=1,2,\dots$



  - with local origin limit = hypergeometric kernel
  - local bulk and edge limits in Ginibre = universal!

today: Unitary group $\int$'s and applications in multi-matrix models (Hermitian or Wishart type: real ev or sv)

* the global density equals the density of a single Ginibre matrix to the power m: $(J)^m = U(Z+T)^m U^\dagger$

$$\Rightarrow \mathrm{Tr}(JJ^\dagger)^m = \sum_{j=1}^{N} z_j^m$$

change of variables: $s = r^m \Leftrightarrow s^{\frac{1}{m}} = r$

$$1 = \int_0^\infty dr\, r\, \rho(r) \cdot 2\pi = \int_0^\infty ds\, \frac{1}{m}\, s^{\frac{1}{m}-1}\, s^{\frac{1}{m}}\, \rho(s^{\frac{1}{m}})\, 2\pi = \int_0^\infty ds\, s\, \tilde{\rho}(s)\, 2\pi$$

e.g. density of single Ginibre — density of J^m

with $\tilde{\rho}(s) = \frac{1}{m}\, s^{\frac{2}{m}-2}$

- Product of m indep Ginibre = (1 Ginibre)^m? for global density of complex ev yes, for local statistics no (for singular values not true for global density either

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Integrals over the unitary group $U(N)$ [Literature math-ph/0209030 hep-th/0002161]

- In the determination of the joint density of eigenvalues (ev) so far the unitary matrices that diagonalise to an ev basis have dropped out $\Rightarrow \int_{U(N)\,\mathrm{Haar}} dU$ = const in the normalisation
- In more general 1- or 2- matrix models this is not always the case
  $\Rightarrow$ I'll present a list of 3 known integrals where the answer is known for arbitrary N

* for integrals over $O(N)$ or $USp(2N)$ this is not true in general!

$\rightarrow$ the technique to be applied (for $U(N)$) is called character expansion

Harish-Chandra (1958) — C. Itzykson, J.B. Zuber (1980)

$$\boxed{\int_{U(N)} [dU] \exp\left[\beta \operatorname{Tr}(XUYU^\dagger)\right] = \text{const}\ \beta^{-\frac{N(N-1)}{2}} \frac{\det\left[e^{\beta x_i y_j}\right]_{i,j=1}^{N}}{\Delta_N(X)\,\Delta_N(Y)}}$$

where $x_1, \dots, x_N$ are the eigenvalues of X (Hermitian)

$y_1 \dots y_N$ are the — " — of Y (— " —)

• the constant depends on the normalisation of the Haar measure

X and Y can be chosen to be $\operatorname{diag}(x_1, \dots, x_N)$ and $\operatorname{diag}(y_1, \dots, y_N)$ without loss of generality, due to the invariance of the Haar measure

let's diagonalise $X = V\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_N\end{pmatrix}V^\dagger$, $Y = W\begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_N\end{pmatrix}W^\dagger$, V, W fixed

$\Rightarrow \operatorname{Tr}(XUYU^\dagger) = \operatorname{Tr}\left(V\begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_N\end{pmatrix}V^\dagger U W\begin{pmatrix} y_1 & & \\ & \ddots & \\ & & y_N\end{pmatrix}W^\dagger U^\dagger\right)$

$\Rightarrow U \to VUW^\dagger$ absorbs the V and W's, and $[dU]$ is invariant!

$$\boxed{\int_{U(N)} [dU] \det(U)^\nu \exp\left[\tfrac{1}{2}\operatorname{Tr}(M^\dagger U + U^\dagger M)\right] = \text{const}\ \frac{\det\left[m_i^{j-1} I_{\nu+j-1}(m_i)\right]_{i,j=1}^{N}}{\Delta_N(M^2)}}$$

where $m_1, \dots, m_N$ are the ev of M

$I_k(x)$ the modified Bessel function of 2nd kind

× a generalisation to $M^\dagger \to A$, $M \to B$ is possible

the above integral goes under several names: Leutwyler-Smilga 1992

Brower, Rossi, Tan (1980)

Berezin - Karpelevich (1958)

$$\int_{U(N)}[dU]\int_{U(M)}[dV]\exp\left[\tfrac{1}{2}\mathrm{tr}(UAVB + B^\dagger V^\dagger A^\dagger U^\dagger)\right] = \mathrm{const}\ \frac{\det I_{N-M}(x_i y_j)}{\prod_{i=1}^{N}(x_i y_i)^{N-M}\Delta_M(x^2)\Delta_M(y^2)}$$

$A, B^\dagger$ are $N\times M$, let's take $M\le N$

where $x^2_{i=1..M}$ are the ev of $A^\dagger A$

$y^2_{j=1..M}$ " " " of $BB^\dagger$

here we integrate over 2 unitary matrices

* the proof for these 3 identities can be found in the above literature)

it uses the so-called character expansion

N.B.: other proofs exist, e.g. by showing that LHS & RHS satisfy the same diffusion type equation.

Idea & ingredients:

- the representations r of $U(N)$ can be labelled by partitions of N into $r=(n_1,n_2,\dots,n_N)$, $n_1\ge\dots\ge n_N$

$\Rightarrow$ the character (= trace of rep. matrix) of the irrep corres to this r reads

$$\chi_{r=(n_1,n_2,\dots,n_N)}(U) = \frac{\det t_i^{n_j+N-j}}{\Delta_N(t_1,\dots,t_N)}$$ Weyl formula, $t_1..t_N$ ev of U

- the integrand can be expanded into such $\chi_r(U)$, e.g.

for $X, Y \in \mathcal{U}(N)$ algebra of $U(N)$ $\quad \exp[\beta\,\mathrm{tr}\,XUYU^\dagger] = \sum_{\substack{\text{all reps } r\\(n_1\ge\dots\ge n_N)}}\alpha_r(\beta)\chi_r(XUYU^\dagger)$

- orthogonality of reps & characters:

$$\int dU\, U^{(r)}_{\alpha\beta}\, U^{*(r')}_{\gamma\delta} = \delta^{rr'}\delta_{\alpha\gamma}\delta_{\beta\delta}\frac{1}{d_r} \Rightarrow \int dU\, e^{\beta\,\mathrm{tr}\,XUYU^\dagger} = \sum_r \frac{\alpha_r}{d_r}\chi_r(X)\chi_r(Y)$$

reps of U $\qquad$ dim of rep r $\qquad = \dots$ HCIZ

Multimatrix models: Application of unitary group integrals

1. Singular values of the product $\boxed{Y_m = J_m \cdots J_1}$ of m Ginibre (Wishart) ens.

singular value decomp:

$N\times N$ complex matrix $\underline{J = U\Lambda V}$, $U, V \in U(N)/U(1)^N$, $\Lambda = \mathrm{diag}(x_1 \ldots x_N)$

with singular values $x_{j=1,\ldots,N} \geq 0$

$\Rightarrow x_j^2 = \mathrm{ev}$ of $JJ^\dagger = U\Lambda^2 U^\dagger$

$m=1$: single ensemble: Wishart - Laguerre

$$e^{-\mathrm{Tr} JJ^\dagger} = e^{-\mathrm{Tr}\, U\Lambda V V^\dagger \Lambda U^\dagger} = e^{-\mathrm{Tr}\Lambda^2}$$

$$C_N = \int_{\mathbb{C}^{N^2}} [dJ]\, e^{-\mathrm{Tr} JJ^\dagger} = \int dU \int dV \prod_{j=1}^N \int_0^\infty dx_j\, x_j\, e^{-x_j^2}\, \Delta_N(x^2)^2$$

here no group integral needed! from Jacobian

$m=2$
$$C_N^{(m=2)} = \int [dJ_1][dJ_2] \exp[-\mathrm{Tr} J_1 J_1^\dagger - \mathrm{Tr} J_2 J_2^\dagger]$$

we seek the jpdf of the product matrix $\boxed{Y_2 = J_2 J_1}$

* for $J \in Gl(N,\mathbb{C})$ invertible (non-invertible ones are of measure zero on all $N\times N$ J's)

$\Rightarrow \boxed{J_2 = Y_2 J_1^{-1}}$, this trafo has a Jacobian $\dfrac{1}{\det[J_1 J_1^\dagger]^N}$

$$\Rightarrow C_N = \int [dJ_1][dY_2]\, e^{-\mathrm{Tr} J_1 J_1^\dagger - \mathrm{Tr}(Y_2 J_1^{-1}(Y_2 J_1^{-1})^\dagger)} \frac{1}{\det[J_1 J_1^\dagger]^N}$$

– strategy: sv decomp of J_1 and Y_2, then integrate out the sv $x_1 \ldots x_N$ of J_1 $\Rightarrow$ jpdf of sv of Y_2

* Here we need the unitary group integral!

- $\boxed{J_1 = U_1 \Lambda_1 V_1} \Rightarrow \mathrm{Tr}\, J_1 J_1^\dagger = \mathrm{Tr}\left(U_1 \Lambda_1 V_1 V_1^\dagger \Lambda_1 U_1^\dagger\right) = \mathrm{Tr}\, \Lambda_1^2$, Jacobian $\prod x_i\, \Delta_N(\Lambda_1^2)^2$

- $\boxed{Y_2 = U_2 \Lambda_2 V_2} \Rightarrow \mathrm{Tr}\left(Y_2 J_1^{-1} J_1^{-1\dagger} Y_2^\dagger\right) = \mathrm{Tr}\left(Y_2^\dagger Y_2 (J_1^\dagger J_1)^{-1}\right)$

$$= \mathrm{Tr}\left(V_2^\dagger \Lambda_2 U_2^\dagger U_2 \Lambda_2 V_2 \left(V_1^\dagger \Lambda_1 \underbrace{U_1^\dagger U_1}_{1} \Lambda_1 V_1\right)^{-1}\right)$$

$$= \mathrm{Tr}\left(V_2^\dagger \Lambda_2^2 V_2 \underbrace{V_1^{-1}}_{V_1^\dagger} \Lambda_1^{-2} \underbrace{V_1^{\dagger -1}}_{V_1}\right) = \mathrm{Tr}\left(V_1 V_2^\dagger \Lambda_2^2 V_2 V_1^\dagger \Lambda_1^{-2}\right)$$

with Jacobian $\sim \Delta_N(\Lambda_2^2)^2 \prod_{j=1}^N y_j$, $\Lambda_2 = \mathrm{diag}(y_1, .., y_N)$

$$\Rightarrow \underline{\underline{C_N^{(m=2)}}} = \int dU_1 \int dV_1 \int dU_2 \int dV_2 \prod_{j=1}^N \int_0^\infty dx_j\, x_j\, e^{-x_j^2} \frac{\Delta_N(x^2)^2}{\prod_{j=1}^N x_j^{2N}} \prod_{\ell=1}^N \int_0^\infty dy_\ell\, y_\ell\, \Delta_N(y^2)^2 \cdot e^{-\mathrm{Tr}\left(V_1 V_2^\dagger \Lambda_2^2 V_2 V_1^\dagger \Lambda_1^{-2}\right)}$$

- the $U_{1,2}$ integrals decouple, and $V_1 \to V_1 V_2$ leaves dV_1, the Haar measure invariant $\Rightarrow V_1$-integral also decoupled

$$\Rightarrow \underline{\underline{C_N^{(m=2)}}} = \mathrm{const} \prod_{j=1}^N \int_0^\infty dx_j\, x_j^{1-2N} e^{-x_j^2} \Delta_N(x^2)^2 \prod_{\ell=1}^N \int_0^\infty dy_\ell\, y_\ell\, \Delta_N(y^2)^2 \underbrace{\int dV_2\, e^{-\mathrm{Tr}\left(\Lambda_2^2 V_2 \Lambda_1^{-2} V_2^\dagger\right)}}_{\text{HCIZ}}$$

$$\overset{\text{HCIZ}}{=} \mathrm{const}\, \frac{\det e^{-y_j^2 \cdot x_\ell^{-2}}}{\Delta_N(y^2)\, \Delta_N(x^{-2})}, \quad \Delta_N(x^{-2}) = \prod_{j>i}^N \left(x_j^{-2} - x_i^{-2}\right) = \prod_{j>i}^N \frac{(x_i^2 - x_j^2)}{x_i^2 x_j^2}$$

$$= (-1)^{\frac{N(N-1)}{2}} \frac{\Delta_N(x^2)}{\prod_{j=1}^N x_j^{2N-2}}$$

⚠ we can cancel many terms, but still need to integrate out the x_j's!

$$\Rightarrow C_N^{(m=2)} = \text{const} \prod_{j=1}^N \int_0^\infty dx_j\, x_j\, e^{-x_j^2} \int_0^\infty dy_j\, y_j\, \Delta_N(x^2)\Delta_N(y^2)\, \det\left[e^{-y_j^2/x_i^2}\right]$$

* the jpdf $P_N(x_1,\dots,x_N; y_1,\dots,y_N)$ is of the form of a classical two matrix model that can be solved with bi-orthogonal polynomials $p_\ell(x), q_j(y)$

$$\boxed{h_j \delta_{jk} = \int_0^\infty dx \int_0^\infty dy\, xy\, e^{-x^2} e^{-y^2/x^2} p_j(x)\, q_k(y)}$$

- integrating out the x_j leads to Meijer G-functions:

substitute $x_j^2 = t_j$, $\underbrace{\int dt_1 \dots dt_N\, e^{-\Sigma t_j}}_{\text{sym in all indices}} \Delta_N(t) \det e^{-y_j^2/t_i} = N! \int dt_1 \dots dt_N\, e^{-\Sigma_j t_j} \Delta_N(t)\, e^{-\Sigma_j y_j^2/t_j}$

$\uparrow$ diagonal

$\Rightarrow$ we can pull the t-integrations into the Vandermonde $\Delta_N(t) = \det\begin{bmatrix} 1 & t_1 & \dots & t_1^{N-1} \\ \vdots & & & \\ 1 & t_N & & t_N^{N-1}\end{bmatrix} \begin{matrix}\leftarrow \int dt_1 \\ \\ \int dt_N\end{matrix}$

recall p. 31 $\quad 2K_\nu(2y) = y^{-\nu} \int_0^\infty dt\, t^{\nu-1} e^{-t-\frac{y^2}{t}}$, $\quad K_{-\nu}(x) = K_\nu(x)$

$$\Rightarrow C_N^{(m=2)} = \text{const} \prod_{j=1}^N \int_0^\infty dy_j\, y_j\, \Delta_N(y^2) \det\Big[\underbrace{2 y_j^{i-1} K_{i-1}(2y_j)}_{\sim G^{20}_{02}}\Big]$$

$\leftarrow$ example for a polynomial ensemble

* this process can be iterated for $Y_m = J_m \cdots J_1 \equiv J_m Y_{m-1}$, where we need to perform $m-1$ unitary group integrals $\Rightarrow$ get G^{m0}_{0m} instead. see e.g. 1303.5694

* the jpdf of sv of Y_m and its kernel can be built up successively, starting with $m=1$ = Wishart Laguerre, then computing the jpdf and kernel (and "OP") for $Y_2 = J_2 J_1$ etc. see e.g. 1404.5802

The classical Hermitian 2-MM : $H_1 = H_1^\dagger$, $H_2 = H_2^\dagger$ $N \times N$

$$Z_N = \int dH_1 dH_2 \exp[-\mathrm{Tr}\, V_1(H_1) - \mathrm{Tr}\, V_2(H_2) + g\, \mathrm{Tr}\, H_1 H_2]$$

Coupling between the 2 matrices

e.g. for $V_1(H_1) = H_1^2$ Gauß. ↑ two indep (G)UE's

for g small enough: convergence

* eigenvalue decomp. $H_1 = U_1 \Lambda_1 U_1^\dagger$, $H_2 = U_2 \Lambda_2 U_2^\dagger$

with $\Lambda_1 = \mathrm{diag}(x_1 \ldots, x_N)$ ev of H_1; Λ_2: $y_1 \ldots y_N$ of H_2

$$\Rightarrow Z_N = \int dU_1 \int dU_2 \prod_{j=1}^{N} \left(\int_{-\infty}^{\infty} dx_j \int_{-\infty}^{\infty} dy_j\, e^{-V_1(x_j)} e^{-V_2(y_j)} \right) \Delta_N(x)^2 \Delta_N(y)^2 \cdot e^{g \mathrm{Tr}\, U_1 \Lambda_1 U_1^\dagger U_2 \Lambda_2 U_2^\dagger}$$

Haar inv $U_2 \to U_1 U_2$

$$\overset{\text{HCIZ integral}}{=} \text{const} \prod_{j=1}^{N} \int_{-\infty}^{\infty} dx_j \int_{-\infty}^{\infty} dy_j\, \underbrace{e^{-V_1(x_j)} e^{-V_2(y_j)} \Delta_N(x) \Delta_N(y) \det[e^{g x_i y_j}]}_{P_N(x_1, \ldots, x_N; y_1, \ldots, y_N)}$$

$w(x_i, y_j)$

$\Rightarrow$ all correlation functions $R_{k,l}(x_1 \ldots, x_k; y_1, \ldots, y_l)$ can be expressed in terms of 4 kernels, containing the bi-orthogonal polynomials $p_l(x)$, $q_j(y)$, that can be constructed from Gram-Schmidt

and their integral transforms $\Psi_l(y) = \int_{-\infty}^{\infty} dx\, w(x, y)\, p_l(x)$

$$\chi_j(x) = \int_{-\infty}^{\infty} dy\, w(x, y)\, q_j(y)$$

see e.g. hep-th/0609059

for examples where p, q are Hermite or Laguerre
