Lecture 2

Summary: non-Hermitian random matrices J

Matrix: $J \neq J^\dagger \left(= J^* = \bar{J}^T\right)$, $J \in \mathbb{C}^{N^2}$

- difficulty to construct invariant and convergent distributions of J_{ij} for $J = U(Z+T)U^\dagger$ non-normal s.th. U, T decouple

examples:
$$P(J) = c \exp\left[-\mathrm{Tr}(JJ^\dagger)\right] \quad \text{Ginibre ensemble}$$
$$P_\tau(J) = c_\tau \exp\left[\frac{-1}{1-\tau^2}\left(\mathrm{Tr}(JJ^\dagger) - \frac{\tau}{2}\left(\mathrm{Tr}(J^2) + \mathrm{Tr}(J^{\dagger 2})\right)\right)\right] \quad \text{elliptic --}$$
$$\tau \in [0,1)$$

Exercise: show that $P_\tau(J)$ interpolates between Ginibre ($\tau=0$) and the GUE ($\tau \to 1$)

Eigenvalues $z_{j=1,\dots,N} \in \mathbb{C}$: for $P(J)$ that satisfy the above, that is we can integrate out U and T we find the

$\beta = 2 \to$
β-ensembles

$$\boxed{P_N(z_1, \dots, z_N) = \prod_{j=1}^{N} w(z_j) \quad |\Delta_N(Z)|^2}$$

joint (probab.) distrib. fuct jpdf of complex ev

with normalisation $Z_N = \prod_{i=1}^{N} \int_{\mathbb{C}} d^2z_i \, w(z_i) \, |\Delta_N|^2$, Δ_N Vandermonde

weight for above examples:
$$w(z) = \exp\left[-|z|^2\right] \quad \text{Ginibre}$$
$$w_\tau(z) = \exp\left[\frac{-1}{1-\tau^2}\left(|z|^2 - \frac{\tau}{2}(z^2 + \bar{z}^2)\right)\right] \quad \text{elliptic --}$$

or rectangular Ginibre matrices J: $N \times (N+\nu)$ $\nu = 0, 1, \dots$

gives [arXiv 1107.5019] $w(z) \to w(z) \cdot |z|^{2\nu}$

* we will see today: $P_N(z_1, \dots, z_N)$ rep a det. point process using OP on $\mathbb{C}$.

recall OP on $\mathbb{C}$ $\quad P_n(z) = z^n + \ldots$ monic w.r.t. weight $w(z)$

$$\boxed{\langle P_n, P_\ell \rangle = \int_{\mathbb{C}} d^2z\, w(z)\, P_n(z)\, \overline{P_\ell(z)} = h_n\, \delta_{n,\ell}}$$ ← Kronecker δ, $h_n = \|P_n\|^2$

Existence: Gram-Schmidt construction

Define the moments $\langle i,j \rangle = \int_{\mathbb{C}} d^2z\, w(z)\, z^i \bar{z}^j$, for $\overline{w(z)} = w(z)$: $\langle i,j \rangle = \langle j,i \rangle$

$$\Rightarrow P_n(z) = \begin{vmatrix} \langle 0,0\rangle & \langle 1,0\rangle & \cdots & \langle n,0\rangle \\ \vdots & \vdots & & \vdots \\ \langle 0,n-1\rangle & \langle 1,n-1\rangle & \cdots & \langle n,n-1\rangle \\ 1 & z^1 & \cdots & z^{n-1} \end{vmatrix} \cdot D_{n-1}^{-1}, \quad D_n = \begin{vmatrix} \langle 0,0\rangle & \cdots & \langle n,0\rangle \\ \vdots & & \\ \langle 0,n\rangle & \cdots & \langle n,n\rangle \end{vmatrix}$$

are the monic OP on $\mathbb{C}$ w.r.t. $w(z)$.

* for the actual construction of $P_n(z)$ this is not always the most practical way, even if this is constructive
* in the literature often when considering OP on $\mathbb{C}$ these are actually taken on curves (e.g. unit circle, see books by B. Simon), that is 1D subset; here: 2D sets or full $\mathbb{C}$

- Properties of OP and their kernel on $\mathbb{C}$.

* For $w(z) = w(|z|)$ the OP are always monomial $P_n(z) = z^n$ (e.g. rot. Ginibre) Proof: Exercise 5.

* using the properties of OP we can compute the normalisation of the jpdf in terms of the squared norms h_j:

$$Z_N = N! \prod_{j=0}^{N-1} h_j$$

Proof for $Z_N = \prod_{j=1}^{N} \int_{\mathbb{C}} d^2z_j\, \omega(z_j)\, |\Delta_N(z)|^2 = \prod_{j=1}^{N}\int_{\mathbb{C}} d^2 z_j\, \omega(z_j)\, \det P_{j-1}(z_k) \det \overline{P_{j-1}(z_k)}$

$$= \sum_{\sigma,\sigma' \in S_N} (-)^{\sigma+\sigma'} \prod_{j=1}^{N} \int_{\mathbb{C}} d^2 z_j\, \omega(z_j)\, \underbrace{P_{\sigma(j)-1}(z_j)\, \overline{P_{\sigma'(j)-1}(z_j)}}_{\delta_{\sigma(j),\sigma'(j)}\, h_{\sigma(j)-1}}$$

$$= \sum_{\sigma \in S_N} \prod_{j=1}^{N} h_{\sigma(j)-1} = \prod_{j=1}^{N} h_{j-1} \underbrace{\sum_{\sigma\in S_N}}_{N!}$$

* in the same way one can prove the following representation

for the monic OP called Heine-formula (on $\mathbb{R}$) proof Exercises 6!

$$\boxed{P_n(z) = \frac{1}{Z_n} \int_{\mathbb{C}} d^2 z_1 \ldots \int_{\mathbb{C}} d^2 z_n \prod_{j=1}^{n} (z - z_j) P_n(z_1 \ldots z_n) = \left\langle \prod_{j=1}^{n} (z - z_j) \right\rangle_n}$$

eigenvalue rep. (check $P_n(z) \sim z^n + \ldots$ for $z \to \infty$) (or $\mathbb{E}\left(\prod_{j=1}^{n}(z-z_j)\right)$)

equally useful is the matrix rep $P_n(z) = \langle \det(z \mathbb{1}_n - J) \rangle_n = \frac{1}{Z_n} \int [dJ] P(J) \det(z - J)$

for J n×n

<u>Def</u> Kernel of OP K_N

$$\boxed{K_N(z, \bar{u}) = \sum_{j=0}^{N-1} \frac{1}{h_j} P_j(z) \overline{P_j(u)}}$$

and it holds $K_N(z, \bar{u}) = \frac{1}{h_{N-1}} \left\langle \prod_{j=1}^{N-1} (z - z_j) \prod_{j=1}^{N-1} (\bar{u} - \bar{z}_j) \right\rangle_{N-1} = \langle \det(z-J) \det(\bar{u} - J^\dagger) \rangle$

which is due to P. Zinn-Justin 98 for $\mathbb{R}$ (cf. GA + G. Vernizzi 2003 on $\mathbb{C}$)

$\sim \begin{vmatrix} P_N(z) & P_{N-1}(z) \\ P_N(u) & P_{N-1}(u) \end{vmatrix} / z-u$

Proof as in, this is $\neq \langle \det(z - J) \det(u - J) \rangle_{N-1}$, what is this?

* obviously it holds $\overline{K_N(z, \bar{u})} = K_N(u, \bar{z})$, $\underline{\int_{\mathbb{C}} d^2 z\, \omega(z)\, K_N(z, \bar{z}) = N}$

and $\underline{\int_{\mathbb{C}} d^2 u\, K_N(z, \bar{u})\, \overline{K_N(u, \bar{v})} = K_N(z, \bar{v})}$

Examples * rectangular Ginibre ensemble $\omega_\nu(z) = |z|^{2\nu} e^{-|z|^2} \overset{Ex5}{\Rightarrow} P_n(z) = z^n$

$$\Rightarrow h_n = \int_{\mathbb{C}} d^2z\, \omega_\nu(z) |P_n(z)|^2 = \int_0^{2\pi} d\varphi \int_0^\infty dr\, r\, r^{2\nu} e^{-r^2} r^{2n} \underset{r^2 = s}{=} 2\pi \frac{1}{2} \int_0^\infty ds\, s^{n+\nu} e^{-s}$$

$$\Leftrightarrow \boxed{h_n = \pi\, \Gamma(n+\nu+1)} \quad \forall n \in \mathbb{N}, \quad \nu > -1 \quad \text{squared norms}$$

with kernel $\boxed{K_N(z,\bar{u}) = \sum_{\ell=0}^{N-1} \frac{z^\ell \bar{u}^\ell}{\pi \Gamma(\ell+\nu+1)}}$

$\leftrightarrow$ incomplete Γ-function:

$\nu = 0$: $\Gamma(\ell+1) = \ell! \Rightarrow K_\infty(z,\bar{u}) = \frac{1}{\pi} e^{z\bar{u}}$

and $\Gamma(N) e^{-z\bar{u}} K_N(z,\bar{u}) = \Gamma(N, z\bar{u})$

where $\Gamma(N,x) = \int_x^\infty dt\, t^{N-1} e^{-t}$

* Note that although $\forall \omega = \omega(|z|)$ the OP $P_n(z)$ are all the same, but the norms and kernels are different in general!

* elliptic Ginibre ensemble $\quad \omega_\tau(z) = e^{-\frac{1}{1-\tau^2}\left(|z|^2 - \frac{\tau}{2}(z^2 + \bar{z}^2)\right)}$

Ex 7: it holds that $\boxed{P_n(z) = \left(\frac{\tau}{2}\right)^{\frac{n}{2}} He_n\left(\frac{z}{\sqrt{2\tau}}\right) = z^n + \ldots}$ are monic OP (He_n: Hermite polynomials $\perp$ on $\mathbb{R}$ wrt $e^{-x^2/2}$, Probab.)

using $H_n(u) = n! \oint_{C_0} \frac{dt}{2\pi i} e^{2ut - t^2} t^{-(n+1)}$ for $n = 0, 1, \ldots$

complex contour integral rep., and $h_n = n!$

* Recall that for OP on $\mathbb{R}$ with general weight a 3-step recurrence relation holds, also for Hermite polynomials.

Why doesn't Christoffel-Darboux hold for the kernel of Hermite as OP on $\mathbb{C}$

$$K_N(z,\bar{u}) = \sum_{\ell=0}^{N-1} \frac{1}{\ell!} \left(\frac{\tau}{2}\right)^\ell He_\ell\left(\frac{z}{\sqrt{2\tau}}\right) He_\ell\left(\frac{\bar{u}}{\sqrt{2\tau}}\right)$$

some Literature: hep-th/9401163, chao-dyn/9802025

• Dyson's Theorem [Thm 5.1.4, Mehta, Random Matrices, 3rd Ed, Elsevier, NY 2004]

For $K_N(z,\bar{u})$ satisfying the conditions on p. 6 ($\overline{K(z,\bar{u})}=K(u,\bar{z})$, $\int K = N$, $\int KK = K$)

it holds
$$\boxed{\int_{\mathbb{C}} d^2 z_L\, w(z_L) \underbrace{\det\left[K_N(z_i,\bar{z}_j)\right]_{i,j=1}^{L}}_{L\times L} = (N-L+1)\underbrace{\det\left[K_N(z_i,\bar{z}_j)\right]_{i,j=1}^{L-1}}_{(L-1)\times(L-1)}}$$

that is a reduction of the size of the det by 1.

The proof uses Laplace expansion, see Mehta (or Exercise)

• DPP: the jpdf on page 4 with general weight is a DPP on $\mathbb{C}$
with kernel of OP on $\mathbb{C}$ wrt this weight:

$$\boxed{P_N(z_1,\dots,z_N) = \prod_{j=1}^N w(z_j)\, |\Delta_N(z)|^2 = \prod_{j=1}^N h_j w(z_j)\, \det\left[K_N(z_i,\bar{z}_j)\right]_{i,j=1}^N}$$

The proof uses simple linear algebra:
$$\text{as } \Delta_N(z) = \det\left[z_i^{j-1}\right]_{i,j=1}^N = \det\left[\underset{\text{monic}}{P_{j-1}(z_i)}\right]_{i,j=1}^N = \prod_{j=1}^N h_{j-1}^{1/2} \det\left[\underset{\text{orthonormal}}{\frac{P_{j-1}(z_i)}{h_{j-1}^{1/2}}}\right]_{i,j=1}^N$$

and $\det A = \det A^T$ we have for $A_{kj} = \dfrac{P_{j-1}(z_k)}{h_{j-1}^{1/2}}$

$$P_N(z_1,\dots,z_N) = \prod_{j=1}^N h_{j-1} w(z_j)\, \det A^T \det \bar{A} = \prod_{j=1}^N h_{j-1} w(z_j)\, \det\Big[\underbrace{\sum_{j=1}^N A_{kj} \bar{A}^T_{ji}}\Big]$$
$$\sum_{j=1}^N \frac{P_{j-1}(z_k)}{h_{j-1}^{1/2}} \frac{\overline{P_{j-1}(z_i)}}{h_{j-1}^{1/2}} = K_N(z_i,\bar{z}_k)$$

$\Rightarrow$ we can now def the properly normalised jpdf and

□

• Complex Eigenvalue Correlation Functions

Def k-point correl function
$$\boxed{R_k(z_1,\dots,z_k) = \frac{N!}{(N-k)!} \frac{1}{Z_N} \int_{\mathbb{C}} d^2 z_{k+1} \dots \int_{\mathbb{C}} d^2 z_N\, P_N(z_1,\dots,z_N)}$$