Lecture 3

Summary:

DPP in the complex plane

$$P_N(z_1,\dots,z_N) = \prod_{j=1}^{N} \omega(z_j)\, |\Delta_N(\{z\})|^2 = \prod_{j=1}^{N}\left(h_{j-1}\,\omega(z_j)\right) \det\left[K_N(z_k,\bar z_l)\right]_{k,l=1}^{N}$$

with kernel $K_N(z,\bar u) = \sum_{\ell=0}^{N-1} \frac{1}{h_\ell} P_\ell(z)\,\overline{P_\ell(u)}$ of OP $P_\ell(z)$ wrt $\omega(z)$

$h_\ell\,\delta_{\ell k} = \int_{\mathbb{C}} d^2z\, \omega(z)\, P_\ell(z)\, \overline{P_k(z)}$.

Dyson's thm $\int_{\mathbb{C}} d^2 z_L\, \omega(z_L) \underset{L\times L}{\det}\left[K_N(z_i,\bar z_j)\right]_{i,j=1}^{L} = (N-L+1) \underset{(L-1)\times(L-1)}{\det}\left[K_N(z_i,\bar z_j)\right]_{i,j=1}^{L-1}$

From Dyson's Thm we immediately have by iteration

$$\boxed{R_k(z_1,\dots,z_k)} = \frac{\cancel{N!}}{(N-k)!}\,\frac{1}{\cancel{N!}\,\pi^{N}\cancel{\prod_{j=0}^{N-1} h_j}}\int_{\mathbb{C}} d^2 z_{k+1}\dots\int_{\mathbb{C}} d^2 z_N \prod_{j=1}^{N}\cancel{h_{j-1}}\,w(z_j)\,\det\left[K_N(z_i,\bar z_j)\right]_{i,j=1}^{N}$$

$$= \frac{1}{(N-k)!}\,(N-N+1)(N-(N-1)+1)\dots(N-(k+1)+1)\prod_{j=1}^{k} w(z_j)\,\det\left[K_N(z_i,\bar z_j)\right]_{i,j=1}^{k}$$

Note that these k-point functions are not normalised, e.g.

- the spectral density = 1-point function $R_1(z_1) = w(z_1)\,K_N(z_1,\bar z_1)$

 has $\int_{\mathbb{C}} d^2 z_1\, R_1(z_1) = N$ (and not 1)

- Example Ginibre: $R_1(z) = e^{-|z|^2}\sum_{l=0}^{N-1}\frac{(z\bar z)^l}{\pi\, l!} = \frac{\Gamma(N,|z|^2)}{\Gamma(N)}$, $N\geq 1$

- 2-point function: $R_2(z_1,z_2) = w(z_1)w(z_2)\begin{vmatrix} K_N(z_1,\bar z_1) & K_N(z_1,\bar z_2) \\ K_N(z_2,\bar z_1) & K_N(z_2,\bar z_2)\end{vmatrix}$

$$= \underbrace{w(z_1)K_N(z_1,\bar z_1)}_{\text{density}}\;\underbrace{w(z_2)K_N(z_2,\bar z_2)}_{\text{ditto}} - \underbrace{w(z_1)w(z_2)\,K_N(z_1,\bar z_2)\,K_N(z_2,\bar z_1)}_{\text{connected part, cluster function}}$$

$\int\int$ gives $N(N-1)$

factorised part

(→ Ursell function / "Mehta")

- N-point: $R_N(z_1,\dots,z_N) = \frac{N!}{Z_N}\,P_N(z_1,\dots,z_N)$, $\left(\int\right)^N$ gives $N!$ ⇒ $\frac{P_N(z_1,\dots,z_N)}{C_N}$ is normalised

Other correlation functions:

- What happens if we integrate out all angles $\varphi_{j=1,\dots,N}$, $z_j = e^{i\varphi_j} r_j$ in polar coordinates?

- distributions of individual eigenvalues, probability that certain areas of $\mathbb{C}$ are void of EV?

• consider a joint density with rotationally invariant weight function,

$w(z) = w(|z|) = w(r)$, $z = r e^{i\varphi}$ $\Rightarrow$ OP $P_n(z) = z^n$

$$\Rightarrow \boxed{P_N(z_1, \dots, z_N) = \prod_{j=1}^{N} w(r_j) \left|\Delta_N(\{z\})\right|^2} \qquad z_j = r_j e^{i\varphi_j} \quad (**)$$

• distribution of the radii: $d^2 z = dr\, r\, d\varphi$, $r \in (0,\infty)$, $\varphi \in [0, 2\pi)$

$$\rho_N(r_1, \dots, r_N) = \prod_{j=1}^{N} \int_0^{2\pi} d\varphi_j \, w(r_j) r_j \left|\Delta_N(\{z\})\right|^2$$

from polar coord $\int dr_j\, r_j$

we exploit that the monomials z^j are OP wrt $w(r)$ due to the angular integration:

$$\Rightarrow \underline{\rho_N(r_1, \dots, r_N)} = \prod_{j=1}^{N} w(r_j) r_j \int_0^{2\pi} d\varphi_j \sum_{\sigma, \sigma' \in S_N} (-)^{\sigma+\sigma'} \prod_{l=1}^{N} z_l^{\sigma(l)-1} \bar{z}_l^{\sigma'(l)-1}$$

and $\displaystyle\int_0^{2\pi} d\varphi\, z^k \bar{z}^l = r^{k+l} \int_0^{2\pi} d\varphi\, e^{i\varphi(k-l)} = r^{k+l}\, 2\pi\, \delta_{kl}$

$$= \prod_{j=1}^{N} w(r_j) r_j \sum_{\sigma \in S_N} \prod_{l=1}^{N} \left(r_l^{2(\sigma(l)-1)}\, 2\pi \right)$$

$$= \boxed{\prod_{j=1}^{N} w(r_j) r_j \ \mathrm{per}\left[r_l^{2(j-1)} \right]_{l,j=1}^{N} (2\pi)^N} \quad \underline{\text{permanent}}$$

example $\mathrm{per}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad + bc$

* unlike the determinant which can be computed polynomially in time depending on N (e.g. by Gauss' elimination) permanents are difficult to compute for N large

• the above statement for $\rho_N(r_1, \dots, r_N)$ is equivalent to state that

Thm [Kostlan]: The set of absolute values $|z_1| = r_1, \dots, |z_N| = r_N$ obtained from $(**)$ has the same distribution as the set of independent random variables $R_1, \dots, R_N$, where the random variable $(R_k)^2$, $k = 1, \dots, N$ has the density

$$\boxed{g_k(y) = y^{k-1} w(\sqrt{y}) \cdot \frac{1}{h_k}} \text{ on } \mathbb{R}_+, \quad h_k = \int_0^\infty dy\, y^{k-1} w(\sqrt{y})$$

Exercise 8: compute the 1- and 2-point distribution of radii

- **Distribution of individual eigenvalues and gap probabilities**

- to understand the formalism we first consider Hermitian RMT, e.g. the GUE or chGUE = Wishart-Laguerre Unitary Ensemble:

$\mathbb{P}(W) \sim e^{-\mathrm{Tr}\, WW^\dagger}$, with ev $\lambda_{i=1,\dots,N} \in \mathbb{R}_+$ of $WW^\dagger$ Hermitian

$W \in \mathbb{C}^{N\times(N+\nu)}$ <=> λ_j are the squared singular values of

$$W = U\Lambda V^\dagger, \quad \Lambda = \mathrm{diag}(\lambda_1,\dots,\lambda_N)$$

with joint density

$$\boxed{P_N(\lambda_1,\dots,\lambda_N) = \prod_{j=1}^{N} w(\lambda_j)\,|\Delta_N(\{\lambda\})|^2}$$

, chGUE: $w(\lambda) = \lambda^\nu e^{-\lambda}$, $\lambda \geq 0$ ($\nu \geq 0$ $N\times(N+\nu)$ rectang.)

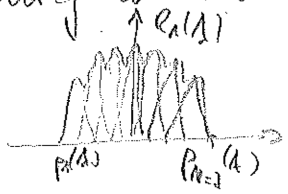
OP: Laguerre polynomials

=> we can order the eigenvalues $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$ and ask, what is the distribution of the smallest ev $p_1(\lambda)$, 2nd smallest $p_2(\lambda)$ etc up to " " largest ev $p_N(\lambda)$, s.t.

$$p_k(\lambda) \geq 0, \quad \int_0^\infty d\lambda\, p_k(\lambda) = 1 \quad \forall\, k = 1,\dots,N$$

* intuitively it is somewhat clear, that knowing all k-point functions R_k is equivalent to knowing all indiv. ev distributions p_k, e.g.

$$R_1(\lambda) = \sum_{l=0}^{N} p_l(\lambda)$$



k-th ev distribution: $k = 1,\dots,N$

$$p_k(s) = k\binom{N}{k} \underbrace{\int_0^s d\lambda_1 \cdots \int_0^s d\lambda_{k-1}}_{\text{absent for } k=1} \underbrace{\int_s^\infty d\lambda_{k+1} \cdots \int_s^\infty d\lambda_N}_{\text{absent for } k=N} \frac{1}{C_N} P_N(\lambda_1,\dots,\lambda_{k-1},\lambda_k = s,\lambda_{k+1},\dots,\lambda_N)$$

notation: absent for $k=1$, absent for $k=N$

why: $\frac{1}{C_N} P_N(\lambda_1, \ldots, \lambda_N)$ is the normalised joint density, the probab to find 1 ev at λ_1, 1 ev at $\lambda_2, \ldots,$ & 1 ev at λ_N. It is symmetric in all $\lambda_{j=1..N}$.

There are $\binom{N}{k}$ possibilities to pick k ev out of N to be the smallest

$\left(=\binom{N}{N-k}\right)$ " " " $N-k$ —— " ——, " " " largest).

and one can check that $\forall k = 1, \ldots, N$ $\quad \underline{\int_0^\infty ds\, p_k(s) = 1}$ $\quad$ Exercise 9.

(explaining the extra factor k)

Exercise 10: compute $p_k(s)$ for LaguerreOP with $\nu = 0$ $\quad w(x) = e^{-x}$

Example $k=1$ smallest ev

$$p_1(s) = N \int_s^\infty d\lambda_2 \ldots \int_s^\infty d\lambda_N \frac{1}{C_N} P_N(\lambda_1 = s, \lambda_2, \ldots, \lambda_N)$$

probab. that all N ev but 1 are $\geq s$ and 1 is at s

$k=N$ largest ev

$$p_N(s) = N \int_0^s d\lambda_1 \ldots \int_0^s d\lambda_N \frac{1}{C_N} P_N(\lambda_1, \ldots, \lambda_{N-1}, \lambda_N = s)$$

probab that all N ev but 1 are $\leq s$ and 1 is at s

* Can we express these $p_k(s)$ in terms of the kernel of OP?
Yes, as a Fredholm determinant, defining the

k-th gap probability (unnormalised)

$$\boxed{E_k(s) = \frac{N!}{(N-k)!} \int_0^s d\lambda_1 \ldots \int_0^s d\lambda_k \int_s^\infty d\lambda_{k+1} \ldots \int_s^\infty d\lambda_N \frac{1}{C_N} P_N(\lambda_1, \ldots, \lambda_N)}$$

for $k = 0, 1, \ldots, N$ (so $\exists$ $N+1$ such quantities):

$\sim$ probability that k ev are $\in [0, s]$ and $N-k$ ev are $\in [s, \infty)$

e.g. <u>0-th gap</u> $\quad E_0(s) = \int_s^\infty d\lambda_1 \ldots \int_s^\infty d\lambda_N \frac{1}{C_N} P_N(\lambda_1 \ldots \lambda_N)$ $\quad$ all ev $\geq s$, $\exists$ gap of ev $[0, s]$

<u>N-th gap</u>: $\quad E_N(s) = N! \int_0^s d\lambda_1 \ldots \int_0^s d\lambda_N \frac{1}{C_N} P_N(\lambda_1, \ldots, \lambda_N)$ $\quad$ all ev $\leq s$, $\exists$ gap of ev $[s, \infty)$