Lecture 4

Summary:

* complex eigenvalue correlation functions

$$\boxed{R_k(z_1,\dots,z_k) = \prod_{l=1}^{k} w(z_l)\, \det\left[K_N(z_i,\bar{z}_j)\right]_{i,j=1}^{k}}$$ given as a $k\times k$ det

in terms of the kernel of OP $K_N(z,\bar{u}) = \sum_{l=0}^{N-1} \frac{P_l(z)\overline{P_l(u)}}{h_l}$

(e.g. density $R_1(z_1) = w(z_1)\, K_N(z_1,\bar{z}_1)$)

* joint density of radii for $w = w(|z|)$, $r_j = |z_j|$ $\forall j = 1,\dots,N$

$$\boxed{\underline{P}_N(r_1,\dots,r_N) = \prod_{l=1}^{N} w(r_l)\ \mathrm{Per}\left[r_i^{2j-1}\right]_{i,j=1}^{N}}$$ radii become indep random var. with different distrib for r_k,

Permanent $\mathrm{Per}[A] = \sum_{\sigma\in S_N} \prod_{l=1}^{N} A_{l\sigma(l)}$

as det, without sign ⇒ is also linear, we can multiply in factors into rows (columns) (e.g. $\prod_{l=1}^{N} r_l\ \mathrm{Per}\left[r_j^{2j-2}\right] = \mathrm{Per}\left[r_i^{2j-1}\right]$)

* distribution of kth individual ev (for Hermitian RMT = real ev !)

$$\boxed{p_k(s) = k\binom{N}{k} \int_0^s d\lambda_1 \dots \int_0^s d\lambda_{k-1} \int_s^\infty d\lambda_{k+1} \dots \int_s^\infty d\lambda_N\, \frac{1}{C_N}\, \underline{P}_N(\lambda_1,\dots,\lambda_{k-1}, s=\lambda_k, \lambda_{k+1},\dots,\lambda_N)}$$

* k-th gap probability

$$\boxed{E_k(s) = \frac{N!}{(N-k)!} \int_0^s d\lambda_1 \dots \int_0^s d\lambda_k \int_s^\infty d\lambda_{k+1} \dots \int_s^\infty d\lambda_N\, \frac{1}{C_N}\, \underline{P}_N(\lambda_1,\dots,\lambda_N)}$$

→ examples

* relation $E_k(s)$ to $p_k(s)$:

$$\partial_s E_0(s) = -p(s)$$ smallest ev distribution

$$\partial_s E_k(s) = k!\,(p_k(s) - p_{k+1}(s)) \quad \text{for } k=1, N-1$$

(also $\partial_s E_N(s) = N!\, p_N(s)$) largest ev distrib.

$\Rightarrow$ telescopic sum $\quad p_k(s) = -\sum_{\ell=0}^{k-1} \frac{1}{\ell!} \partial_s E_\ell(s)$ for $k=1,\dots N$

<u>Exercise 11</u>: proof

* <u>not all $E_k(s)$ are indep</u> : $1-E_0(s) \sim$ probab that at least 1 ev $< s$ can be expressed through all $E_1(s), \dots, E_N(s)$

• <u>Relation between k-point correlation functions and k-th gap probab.:</u>

idea: $(a-b)^j = \sum_{\ell=0}^{j} \binom{j}{\ell} a^{j-\ell}(-b)^\ell$, apply to $\int_s^\infty = \int_0^\infty - \int_0^s$

using the <u>symmetry</u> of $P_N(\lambda_1, \dots \lambda_N)$ under relabelling

$$\Rightarrow E_k(s) = \frac{N!}{(N-k)!} \int_0^s d\lambda_1 \dots \int_0^s d\lambda_k \underbrace{\int_s^\infty d\lambda_{k+1} \dots \int_s^\infty d\lambda_N}_{= \sum_{\ell=0}^{N-k} (-1)^\ell \binom{N-k}{\ell} \left(\int_0^\infty\right)^{N-k-\ell} \left(\int_0^s\right)^\ell d\lambda_{k+1} \dots d\lambda_N} \frac{1}{C_N} P_N(\lambda_1, \dots \lambda_N)$$

$$\binom{N-k}{\ell} = \frac{(N-k)!}{(N-k-\ell)!\,\ell!}$$

using the definition of the $(k+\ell)$-point correl funct. $R_{k+\ell}$ given as $N-(k+\ell)$ integral $\int_0^\infty$ over $\frac{1}{C_N} P_N(\lambda_1..\lambda_N)$

we have
$$\boxed{E_k(s) = \sum_{\ell=0}^{N-k} \frac{(-1)^\ell}{\ell!} \int_0^s d\lambda_1 \dots \int_0^s d\lambda_{k+\ell}\, \underbrace{R_{k+\ell}(\lambda_1, \dots \lambda_{k+\ell})}_{= \det K_N(\lambda_i, \lambda_j)_{(k+\ell)\times(k+\ell)} \prod^{k+\ell} \omega(\lambda)}}$$

* e.g. for $k=0$ $\quad E_0(s) = \sum_{\ell=0}^{N} \frac{(-1)^\ell}{\ell!} \int_0^s d\lambda_1 \dots \int_0^s d\lambda_\ell \det[K_N(\lambda_i, \lambda_j)]_{i,j=1}^{\ell} \prod_{m=1}^{\ell} \omega(\lambda_m)$

where the $\ell=0$ - term is 1

this is the definition of the Fredholm determinant for the kernel K_N

as an integral operator on $f \in L^2(\tilde{\omega})$:

$$(K_N f)(x) = \int_0^\infty dy\, \tilde{\omega}(y) K_N(x,y) f(y), \quad \text{with } \tilde{\omega}(y) = \omega(y)\mathbb{1}_{[0,s]}(y) \text{ here}$$

$$E_0(s) = \det\left(1 - K_N \,|\, L^2(\tilde{\omega})\right)$$

proof: see chapter 4 in Oxford handbook on RMT

* in the example with $k=0$ (on p(3)) we have defined the zeroth order term as unity:

$$E_0(s) = 1 - \int_0^s d\lambda_1 R_1(\lambda) + \frac{1}{2}\int_0^s d\lambda_1 \int_0^s d\lambda_2\, R_2(\lambda_1,\lambda_2) - \frac{1}{3!}\int_0^s d\lambda_1 \int_0^s d\lambda_2 \int_0^s d\lambda_3\, R_3(\lambda_1,\lambda_2,\lambda_3) + \dots$$

* for some considerations, e.g. the smallest ev distrib., this sum sometimes "converges" very rapidly, see hep-th/0311171

• generating function:

define $$E(s;\xi) = 1 + \sum_{\ell=1}^{N} (-\xi)^\ell \frac{1}{\ell!}\int_0^s d\lambda_1 \dots \int_0^s d\lambda_\ell\, R_\ell(\lambda_1,\dots,\lambda_\ell)$$

it holds that $$\boxed{E_k(s) = (-1)^k \frac{\partial^k}{\partial \xi^k} E(s;\xi)\Big|_{\xi=1}} \quad \text{for } k=0,1,\dots,N$$

(where $k=0$ has no derivative). Exercise 12 proof

Q: How can we generalise this concept to ev in the complex plane $\mathbb{C}$, where there is no unique ordering procedure?

→ radial ordering of ev z_j w.r.t. their modulus $r_j = |z_j|$

But: 2 ev can have $r = |z_j| = |z_{k\neq j}|$  without penalty,

→ ∃ level repulsion?

[See e.g. 0901.0897] other convention $\frac{1}{k!}$

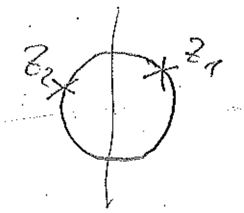
- <u>Distribution of individual ev and gap probabilities on $\mathbb{C}$</u>

On $\mathbb{C}$ there is no unique ordering for the eigenvalues

$\rightarrow$ we will proceed with <u>radial ordering</u>

s/th. $|z_1| \leq |z_2| \leq \ldots \leq |z_N|$

* clearly, unlike for real ev the Coulomb repulsion is not necessarily felt for 2 ev with the same radii $|z_1| = |z_2|$, unless $|z_1 - z_2|$ is small! (we have seen that considering only radii these become indep.)



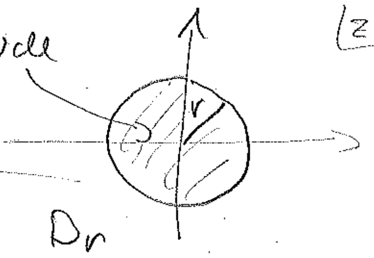
* other orderings are possible (and more useful) e.g. for the elliptic Ginibre close to the GUE w/ some family of contours, e.g.

Radial gap probability

$$\boxed{E_k(r) = \frac{N!}{(N-k)!} \int_0^r dr_1 r_1 \cdots \int_0^r dr_k r_k \int_r^\infty dr_{k+1} r_{k+1} \cdots \int_r^\infty dr_N r_N \underbrace{\prod_{l=1}^{N} \int_0^{2\pi} d\theta_l}_{\mathbb{Z}} \cancel{w(z_l)} \frac{1}{C_N} P_N(z_1, \ldots, z_N)}$$

all ev integrated out

$\sim$ probability to have k ev inside and $N-k$ outside of the disk around 0 of <u>radius r</u> D_r



e.g. $E_0(r)$ prob that this circle is empty

$$\Rightarrow \quad \boxed{\frac{\partial}{\partial r} E_k(r) = r\left(p_k(r) - p_{k+1}(r)\right)} \qquad (\text{set } p_0(r) \equiv 0)$$

where $p_k(r)$ is the radial distribution of the k-th ev in radial <u>ordering</u>. Ex to write down the def of $p_k(r)$

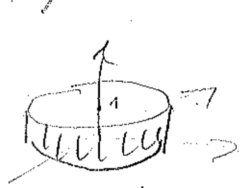
$$\Rightarrow \quad p_k(r) = -\frac{1}{r}\frac{\partial}{\partial r}\sum_{\ell=0}^{k-1} E_\ell(r)$$

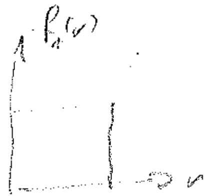
* relation to the density $R_1(z_1)$ $\quad \left(N=\int d^2z\, R_1(z) = \int_0^\infty dr\, r \int_0^{2\pi} d\varphi\, R_1(re^{i\varphi})\right)$

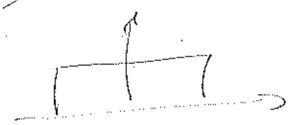
$$\int_0^{2\pi} d\varphi\, R_1(re^{i\varphi}) = \sum_{k=1}^{N} p_k(r), \quad \text{with} \quad \int_0^\infty dr\, r\, p_k(r) = 1 \quad \forall k=1,\dots,N$$

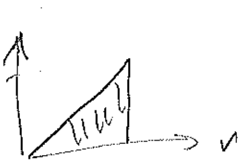
Note that we could also call $\tilde{p}_k(r) = r\, p_k(r)$ the radial distrib. w.r.t. flat measure, and

$$\tilde{R}_1(r) = r\int_0^{2\pi} d\varphi\, R_1(re^{i\varphi}) \quad \text{the radial density}$$

e.g. for Ginibre 2D  $R_1(z)$ $\Rightarrow \int_0^{2\pi} d\varphi\, R_1(re^{i\varphi}) = 2\pi R_1(r)$



(or 1D and )

But $\tilde{R}_1(r)$ 

"triangular law"

Fredholm determinant representation:

as for ev $\in \mathbb{R}$ we can show that the def of E_k ws is equivalent to

1st term in Σ is 1

$$\boxed{E_k(r) = \sum_{\ell=0}^{N-k} \frac{(-1)^\ell}{\ell!} \prod_{j=1}^{k+\ell} \int_0^r dr_j\, r_j \int_0^{2\pi} d\varphi_j\, R_{k+\ell}(z_1,\dots,z_{k+\ell})}\ , \quad k=0,1,\dots,N$$

and a gen. funct

$$E_N(r;\xi) = \qquad (1-\xi)^\ell \qquad \Rightarrow \quad E_k(r) = (-1)^k \frac{\partial^k}{\partial\xi^k} E(r;\xi)\Big|_{\xi=1}$$

alternatively we can write with $\Psi_j(z) = w(z)^{\frac{1}{2}}\frac{P_j(z)}{h_j^{\frac{1}{2}}}$ ON "wave" function from OP

$$E_0(r) = \frac{1}{c_N}\prod_{\ell=1}^{N}\int_{\mathbb{C}\setminus D_r} d^2z_\ell\, w(z_\ell)\, |\Delta_N(\{z\})|^2 = \frac{1}{N!}\prod_{\ell=1}^{N}\int_{\mathbb{C}\setminus D_r} d^2z_\ell\, \left|\det\left[\Psi_{j-1}(z_i)\right]_{i,j=1}^{N}\right|^2$$