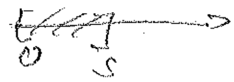
Lecture 5

Summary

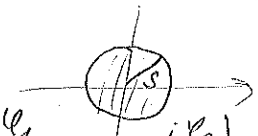
generating functional = Fredholm determinant for k-th gap on $\mathbb{R}$

$$\boxed{E(s;z) = 1 + \sum_{\ell=1}^{N} \frac{(-z)^\ell}{\ell!} \int_0^s d\lambda_1 \dots \int_0^s d\lambda_\ell \, R_\ell(\lambda_1, \dots, \lambda_\ell)}$$

with $E_k(s) = (-1)^k \frac{\partial^k}{\partial z^k} E(s;z)\Big|_{z=1}$



analogous quantity on $\mathbb{C}$ - gap wrt radial domain



$$E(s;z) = 1 + \sum_{\ell=1}^{N} \frac{(-z)^\ell}{\ell!} \int_0^s dr_1 r_1 \dots \int_0^s dr_\ell r_\ell \prod_{j=1}^{\ell} \int_0^{2\pi} d\varphi_j \, R_\ell(r_1 e^{i\varphi_1}, \dots, r_\ell e^{i\varphi_\ell}) =$$

with the same relation to $E_k(s)$ kth gap on $\mathbb{C}$

$\Rightarrow$ individual eigenvalue distributions w.r.t. radial ordering

* repeat: Andréief integral identity

Andreief - integration formula (19th century) <u>Exercise 14: Proof</u>

$$\boxed{\int d\mu(x_1) \ldots \int d\mu(x_N) \det[\phi_i(x_j)]_{i,j=1}^{N} \det[\psi_i(x_j)]_{i,j=1}^{N} = N! \det\left[\int d\mu(x)\phi_i(x)\psi_j(x)\right]_{i,j=1}^{N}}$$

for any measurable functions ϕ_i, ψ_j on any domain $\subseteq \mathbb{R}, \mathbb{C}$ s.th. all int. exist.

The proof merely uses Laplace expansion

$$\sum_{\sigma,\sigma' \in S_N} (-)^{\sigma+\sigma'} \prod_{\ell=1}^{N} \int d\mu(x_{\sigma(\ell)}) \underbrace{\phi_{\sigma(\ell)}(x_\ell)}_{\phi_{\sigma(\sigma'(\ell))}(x_{\sigma'(\ell)})} \psi_\ell(x_{\sigma'(\ell)}) = N! \sum_{\underset{\sigma(\sigma')}{\sigma'' \in S_N}} (-)^{\sigma''} \prod_{\ell=1}^{N} \int dx\, \phi_{\sigma(\ell)}(x)\psi_\ell(x)$$

$$\Rightarrow E_0(v) = \det\left[\int_{\mathbb{C}\setminus D_r} d^2z\, \psi_{j-1}(z)\overline{\psi_{k-1}(z)}\right]_{j,k=1}^{N} = \det\left[\delta_{jk} - \int_{D_r} d^2z\, \psi_{j-1}(z)\overline{\psi_{k-1}(z)}\right]_{j,k=1}^{N}$$

these are orthonormal on $\mathbb{C}$

which is the Fredholm determinant for the integration domain D_r

* we want to diagonalise the ~~determinant~~ matrix inside the and find its eigenvalues

$$E_0(v) = \prod_{j=1}^{N}(1-f_j) \qquad \text{where the } f_j \text{ are the ev of the}$$

integral eq

$$\lambda\psi(u) = \int_{D_r} d^2z\, w(z) \underbrace{\sum_{j=0}^{N-1} \frac{P_j(u)\overline{P_j(z)}}{h_j}}_{= \sum_{j=0}^{N-1} \psi_j(u)\overline{\psi_j(z)}} \psi(z)$$

- the same steps can be repeated for the generating function

$$E(v;\xi) = \ldots = \det\left[\delta_{jk} - \xi \int_{D_r} d^2z\, w(z) \frac{P_{j-1}(z)\overline{P_{k-1}(z)}}{h_{j-1}^{\frac12} h_{k-1}^{\frac12}}\right] = \prod_{j=1}^{N}(1-\xi f_j)$$

$\Rightarrow$ all k-th gaps, e.g. $\qquad \exists!$ ev in D_r

$$\underline{k=1}: \boxed{E_1(v) = -\partial_\xi E_0(v;\xi)\big|_{\xi=0} = \prod_{j=1}^{N}(1-f_j)\sum_{k=1}^{N}\frac{f_k}{1-f_k}} \quad \text{etc.}$$

• Example Ginibre: $w(z) = e^{-|z|^2}$, $h_k = \pi k!$, $P_\ell(z) = z^\ell$

$$\Rightarrow \int_{D_r} d^2z\, \psi_{j-1}(z)\overline{\psi_{k-1}(z)} = \int_0^r dt\, t \int_0^{2\pi} d\varphi\, e^{-t^2} \frac{(te^{i\varphi})^{j-1}(te^{-i\varphi})^{k-1}}{\sqrt{(\pi(j-1)!)(\pi(k-1)!)}}, \qquad z = te^{i\varphi}$$

$$= \frac{\delta_{jk}}{(j-1)!}\, 2\pi \int_0^r dt\, t^{2j-1} e^{-t^2} = \delta_{jk} f_j \text{ is already diagonal! true } \forall \text{ weights } w = w(r)$$

$$\overset{t^2=s}{\Longrightarrow}\quad 1 - f_j = \frac{\int_0^\infty ds\, s^{j-1} e^{-s} - \int_0^{r^2} ds\, s^{j-1} e^{-s}}{(j-1)!} = \frac{1}{\Gamma(j)} \int_{r^2}^\infty ds\, s^{j-1} e^{-s} \underset{\text{see p.7}}{=} \frac{\Gamma(j, r^2)}{\Gamma(j)}$$

for $j = 1, \dots, N$

$$= e^{-r^2} \sum_{\ell=0}^{j-1} \frac{r^{2\ell}}{\ell!}$$

$$\Rightarrow \boxed{E_0(r) = \prod_{j=1}^{N} \frac{\Gamma(j, r^2)}{\Gamma(j)}}$$

, is normalised as $\frac{\Gamma(j, r^2=0)}{\Gamma(j)} = 1 \; \forall j$

$$\Rightarrow \boxed{p_1(r) = -r\partial_r E_0(r) = \sum_{k=1}^{N} \frac{2r^{2k} e^{-r^2}}{\Gamma(k)} \prod_{\substack{j=1 \\ j\neq k}}^{N} \frac{\Gamma(j, r^2)}{\Gamma(j)}}$$

the distribution of the smallest eigenvalue in radius

* Other correlation functions - the spacing distribution:

outlook: for the 3 classical Wigner-Dyson ensembles the spacing distribution, the probability that two consecutive EV have distance s, can be computed analytically for $N = 2$: for $\beta = 1, 2, 4$ (GOE/GUE/GSE)

$$\boxed{p^\beta_{N=2}(s) = a_\beta s^\beta e^{-b_\beta s^2}}$$

Exercise: compute a_β and b_β $\forall \beta > 0$

where a_β, b_β are normalisation constants that follow from

$$\boxed{1 = \int_0^\infty ds\, p^\beta(s) = \int_0^\infty ds\, s\, p^\beta(s)}$$

(Wigner surmise)

* it turns out that this is an excellent approximation to $p^\beta(s)$ in the limit $N \to \infty$ in the bulk of the spectrum. The exact result is a Fredholm det of the sine-kernel (or $\prod$ of spheroidal funct, see Mehta's book)

and the spacing distribution is maybe not as unambiguously defined. We will choose the following def:

$p(s)$: <u>Spacing distrib.</u>: given that there is an ev at $z_0 \in \mathbb{C}$, what is the probability that in radial distance $s = |z - z_0|$ the next ev is found at distance s? Its derivative gives: $p(s)$

* For the Ginibre ensemble (and in principle others) we can easily compute a closed form expression for any N of the following: <u>Spacing distrib. at the origin</u>: given an ev at $z_0 = 0$, what is the probab. to find the next ev at radius s?

$\Rightarrow$ because (for Ginibre (and others)) there is translation invariance in the bulk at large N, this is expected to be true at $z_0 \neq 0$ too

$$\underline{\mathbb{P}(z_1 = 0, |z_2|, \ldots, |z|_N \geq s)} = \int_s^\infty dr_2 r_2 \ldots \int_s^\infty dr_N r_N \prod_{\ell=1}^{N} \int_0^{2\pi} d\varphi_\ell \frac{1}{c_N} P_N(z_1 = 0, z_2, \ldots, z_N)$$

$$= \frac{1}{N! \prod_{j=0}^{N-1} h_j} \int_s^\infty dr_2 r_2 \ldots \int_s^\infty dr_N r_N \int_0^{2\pi} d\varphi_1 \, \omega(z_1 = 0) \prod_{\ell=2}^{N} \int_0^{2\pi} d\varphi_\ell \, \omega(z_\ell) \left| \prod_{j > i \geq 2} (z_j - z_i) \right| \prod_{k=2}^{N} |z_k - 0|^2$$

$\Rightarrow$ this is proportional to the gap probability $E_0(s)$ for $N-1$ ev with modified weight $\omega(z) \to |z|^2 \omega(z) \;\hat{=}\; \nu = 1$ in ω_ν

* <u>example Ginibre</u>: $e^{-|z|^2} \to |z|^2 e^{-|z|^2} \;\hat{=}\;$ rectangular Ginibre with 1 zeromode ✓

$$\Rightarrow h_j^{\nu=1} = 2\pi \int_0^r dt \, t^2 e^{-t^2} t^{2j-2} = h_{j+1}$$

$$\Rightarrow \boxed{\mathbb{P}(z_1 = 0, |z_2|, \ldots, |z_N| \geq r) = \prod_{j=1}^{N-1} \frac{\Gamma(j+1, r^2)}{\Gamma(j+1)} = E_0^{(\nu=1)}(r) \Big|_{N-1}}$$

$\Rightarrow$ spacing distrib. at the origin

$$p(r) = -\partial_r E_0^{r=1}(r) = \sum_{k=1}^{N-1} \frac{1}{\Gamma(k+1)} \left(2 r r^{2k} e^{-r^2}\right) \prod_{\substack{j=1 \\ j\neq k}}^{N-1} \frac{\Gamma(j+1, r^2)}{\Gamma(j+1)}$$

$$\underset{r\to 0}{\sim} r^3 + O(r^5)$$ Exercise 8: show this

* the repulsion at small distance $r \to 0$ is cubic compared to quadratic for the GUE and thus weaker. This cubic behaviour seems to be universal for other distributions, cf cond-mat/9610073.

* Note that $N=2$ is not a good approximation for the large-N result (cf. 0907.4195)

[Lit. 1208.0187]

diagr. 2   Products of Gaussian random matrices

consider $J_i \in \mathbb{C}^{N\times N}$ with $i = 1, 2, \ldots, m$, with

distribution $P(J_i) = c \, \exp[-\mathrm{Tr}\, J_i J_i^+]$

Q: What is the distribution of complex eigenvalues $z_1, \ldots, z_N$ of the product matrix $\boxed{Y_m = J_1 J_2 \ldots J_m} \neq Y_m^+$?

- the same Q is much more difficult when multiplying GUE matrixes!
- we could also make every factor rectangular

in analogy we define the partition function $C_N^{(m)}$

$$C_N^{(m)} \sim \prod_{i=1}^{m} \int [dJ_i] \, e^{-\mathrm{Tr}(J_i J_i^+)}$$

A: the resulting joint density of $z_{1 \ldots N}$ is a OPP:

$$\boxed{P_N^{(m)}(z_1, \ldots, z_N) = \prod_{j=1}^{N} w_m(z_j) \, |\Delta_N(\{z\})|^2}$$

with weight $w_m(z) = \pi^{m-1} \, G^{m\,0}_{0\,m}\left(\underbrace{\overline{0, \ldots, 0}}_{m} \Big| |z|^2 \right)$

given by a Meijer G-function $G^{mn}_{pq}\left({a_1 \ldots a_p \atop b_1 \ldots b_q} \Big| x \right)$

→ in order to understand this result we need to understand 2 ingredients:

* a generalized Schur decomposition ⇒ Jacobian
* how does the Meijer-G arise?

⇒ in the next step we need to compute the norms $h_j^{(m)} \sim (j!)^m$ ⇒ Kernel