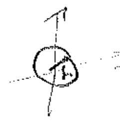
Summary Lecture 6

From Andreief integration formula

$$\int d\mu(x_1)\dots\int d\mu(x_N)\,\det_{1\le i,j\le N}[\phi_i(x_j)]\,\det_{1\le i,j\le N}[\psi_i(x_j)] = N!\,\det_{1\le i,j\le N}\left[\int d\mu(x)\,\phi_i(x)\,\psi_j(x)\right]$$

we obtained radial gap probab. & individual ev distributions:

e.g. in Ginibre $E_0(r) = \prod_{j=1}^{N} \frac{\Gamma(j,r^2)}{\Gamma(j)}$ probab that no ev in disk D_r



$\Rightarrow \rho_1(r) = \dots$ etc

spacing distributions:

for hermitian RMT with real ev $p^{(\beta)}(s) \sim s^\beta e^{-b_\beta s^2}$ Wigner Surmise

$\sim s^\beta \dots$

for non –"– –"– with complex ev

in Ginibre, at origin $P(1\,\text{ev at } z=0, \text{within } r) = \prod_{j=1}^{N-1} \frac{\Gamma(j+1,r^2)}{\Gamma(j+1)}$

$\sim r^3 \dots$

Products of Ginibre-matrices

* are determinantal point process with new weight

[arXiv:1208.0187]

chapter 2 Products of Gaussian random matrices

consider $J_i \in \mathbb{C}^{N\times N}$ with $i = 1, 2, \ldots, m$, with

distribution $P(J_i) = c \exp[-\mathrm{Tr}\, J_i J_i^+]$

Q: What is the distribution of complex eigenvalues $z_1, \ldots, z_N$ of the

product matrix $\boxed{Y_m = J_1 J_2 \ldots J_m} \neq Y_m^+$?

- the same Q is much more difficult when multiplying GUE matrices!
- we could also make every factor rectangular

in analogy we define the partition function $C_N^{(m)}$

$$C_N^{(m)} \sim \prod_{i=1}^{m} \int [dJ_i]\, e^{-\mathrm{Tr}(J_i J_i^+)}$$

A: the resulting joint density of $z_{1\ldots N}$ is a OPP:

$$\boxed{P_N^{(m)}(z_1, \ldots, z_N) = \prod_{j=1}^{N} w_m(z_j)\, |\Delta_N(\{z\})|^2}$$

with weight $w_m(z) = \pi^{m-1} G^{m\,0}_{0\,m}\left(\underbrace{0, \ldots, 0}_{m} \middle| |z|^2\right)$, ~~has moments~~

given by a Meijer G-function $G^{mn}_{pq}\left({a_1 \ldots a_p \atop b_1 \ldots b_q} \middle| x\right)$

→ in order to understand this result we need to understand 2 ingredients:

* a generalized Schur decomposition ⇒ Jacobian
* how does the Meijer-G arise?

⇒ in the next step we need to compute the norms $h_j^{(m)} \sim (j!)^m$ ⇒ kernel

(→ recall standard Schur decomp first!)

Generalized Schur decomposition

[Lit: Matrix Computations by G.H. Golub, C.F. van Loan Johns Hopkins Univ. Press, 1989 → chapter 7.7]

Let $A, B \in \mathbb{C}^{N\times N}$ ($GL(n,\mathbb{C})$, $\exists B^{-1}$)

solutions to $\boxed{0 = \det(\underbrace{A - zB}_{\text{("pencil")}})} = \det(AB^{-1} - z\mathbb{1}_N)\det(B)$

= generalisation of the solution to the characteristic eq. with $B = \mathbb{1}_N$.

Thm 7.7.1. in Golub, v. Loan states the gen. Schur decomp for 2 matrices:

$\exists$ 2 unitary matrices $U, V \in U(N)$ and 2 upper triangular complex matrices S, T such that

$$* \quad \boxed{A = UTV^+ \quad , \quad B = USV^+} \Rightarrow B^{-1} = (USV^+)^{-1} = V^{+-1}S^{-1}U^{-1} = VS^{-1}U^+$$

$\Rightarrow \det(A - zB) = \det(U(T - zS)V^+) = \prod_{i=1}^{N}(T_{ii} - zS_{ii})(\det(UV^+))$

and in particular

$AB^{-1} = UTV^+VS^{-1}U^+ = UTS^{-1}U^+$ has a Schur decomp (Exercise N: S^{-1} is also upper triangular)

and $B^{-1}A = VS^{-1}U^+UTV^+ = VS^{-1}TV^+$ holds for the Schur decomp. of the product matrix

$\Rightarrow$ the complex eigenvalues $z_1, \dots, z_N$ of AB^{-1} (and $B^{-1}A$) are given by $z_i = (TS^{-1})_{ii} = \sum_{j=1}^{N} \underset{i\le j}{T_{ij}} \underset{j\le i}{S^{-1}_{ji}} = T_{ii}S^{-1}_{ii}$

Note: the complex ev of A are not T_{ii} (this $*$ is not the standard Schur decomp of A and B)
— " — B^{-1} — " — S^{-1}_{ii}

The proof uses that any matrix $B \in \mathbb{C}^{N\times N}$ can be approx by a sequence of invertible ($GL(n,\mathbb{C})$) matrices and the QR decomposition for general $A \in \mathbb{C}^{N\times N}$: $A = QR$, $Q \in U(N)$ and R upper triangular complex

Exercise 17: Lemma generalised Schur for products of $GL(N,\mathbb{C})$

For any finite product of $m \geq 2$ matrices A_j the generalised Schur decomposition holds, that is there exist unitary matrices $U_{j=1,\dots,m}$ and upper triangular matrices T_j such that it holds $A_j = U_j T_j U_{j+1}^\dagger$, $j = 1,\dots,m$ with $U_{m+1} = U_1$

$$\Rightarrow \quad Y_m = A_1 A_2 \dots A_m = U_1 T_1 U_2^\dagger U_2 T_2 U_3^\dagger \dots U_m T_m U_1^\dagger$$
$$= U_1 (T_1 \cdots T_m) U_1^\dagger$$

upper triangular, the diagonal elements contain the complex eigenvalues $z_i \overset{\text{Exercise 18}}{=} (T_1)_{ii} (T_2)_{ii} \dots (T_m)_{ii}$, $i = 1..N$

- change of variables and Jacobian [Example: 1208.0187, general proof 1308.6817]

Example $m = 3$:

we want to know the complex eigenvalues of $Y_{m=3} = J_1 J_2 J_3$.

- From above we know that $J_3 J_1 J_2$ and $J_2 J_3 J_1$ have the same complex eigenvalues (why?)
- consider matrix $B = \begin{pmatrix} 0 & J_1 & 0 \\ 0 & 0 & J_2 \\ J_3 & 0 & 0 \end{pmatrix} \Rightarrow B^3 = \begin{pmatrix} J_1 J_2 J_3 & 0 & 0 \\ 0 & J_2 J_3 J_1 & 0 \\ 0 & 0 & J_3 J_1 J_2 \end{pmatrix}$

has the same eigenvalues as $Y_{m=3}$ but $m = 3$ fold degenerate

$\Rightarrow$ we can "diagonalise" B by the following unitary trafo $\mathcal{U}$ based on the gen. Schur decompositions

$$J_1 = U_1 T_1 U_2^\dagger, \quad J_2 = U_2 T_2 U_3^\dagger, \quad J_3 = U_3 T_3 U_1^\dagger$$

as
$$B = \begin{pmatrix} 0 & J_1 & 0 \\ 0 & 0 & J_2 \\ J_3 & 0 & 0 \end{pmatrix} = \underbrace{\begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & U_3 \end{pmatrix}}_{\mathcal{U}} \begin{pmatrix} 0 & T_1 & 0 \\ 0 & 0 & T_2 \\ T_3 & 0 & 0 \end{pmatrix} \underbrace{\begin{pmatrix} U_1^\dagger & 0 & 0 \\ 0 & U_2^\dagger & 0 \\ 0 & 0 & U_3^\dagger \end{pmatrix}}_{\mathcal{U}^\dagger}$$

[Lit.]
$\Rightarrow$ the Jacobian of this transformation is (for general m)

$$|\Delta_N(\{z\})|^2 = \left|\prod_{j>i}^{N} (z_j - z_i)\right|^2 = \left|\prod_{j>i} \left((T_1)_{jj}(T_2)_{jj}\cdots(T_m)_{jj} - (T_1)_{ii}(T_2)_{ii}\cdots(T_m)_{ii}\right)\right|^2$$

* in order to obtain the jpdf of $z_1, \ldots, z_N$ we still need to integrate out the U_j <u>and</u> $(T_j)_{k<l}$, the strictly upper triangular parts

Define $T_j = Z_j + D_j$ for the gen. Schur decomp, $j = 1, \ldots, m$

$$Z_j = \begin{pmatrix} z_1^{(j)} & & 0 \\ & \ddots & \\ 0 & & z_N^{(j)} \end{pmatrix}, \quad D_j = \begin{pmatrix} 0 & & \boxed{\ast} \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \quad \Rightarrow z_i = z_i^{(1)} \cdots z_i^{(m)} \text{ are the complex ev.}$$

* we have seen on page 3:

$$\forall j = 1, \ldots, m \quad \mathrm{Tr}(J_j J_j^\dagger) = \mathrm{Tr}\left(U_j (Z_j + D_j) U_{j+1}^\dagger U_{j+1} (Z_j^\dagger + D_j^\dagger) U_j^\dagger\right)$$
$$= \mathrm{Tr}(Z_j Z_j^\dagger) + \mathrm{Tr}(D_j D_j^\dagger) = \sum_{i=1}^{N} |z_i^{(j)}|^2 + \mathrm{Tr}(D_j D_j^\dagger)$$

$\Rightarrow$ in each factor $\exp[-\mathrm{Tr}(J_j J_j^\dagger)]$ the two factorise and we have

For the partition function

$$G_N^{(m)} \propto \prod_{j=1}^{m} \int [dY_j]\, e^{-\mathrm{Tr}(Y_j Y_j^\dagger)}$$

$$= \prod_{j=1}^{m} \left\{ \int_{\text{Haar}} [dU_j] \int [dQ_j]\, e^{-\mathrm{Tr}(Q_j Q_j^\dagger)} \right\} \int_{\mathbb{C}} dz_1^{(1)} \cdots dz_N^{(1)}\, e^{-\sum_{i=1}^{N} |z_i^{(1)}|^2} \cdots$$

this gives a const.

$$\cdot \int_{\mathbb{C}} dz_1^{(m)} \cdots dz_N^{(m)}\, e^{-\sum_{i=1}^{N} |z_i^{(m)}|^2}\, |\Delta_N(\{z\})|^2 \prod_{j=1}^{N} \delta(z_j - z_j^{(1)} \cdots z_j^{(m)})$$

this contains already the complex ev $z_i = z_i^{(1)} \cdots z_i^{(m)}$ of Y_m
$i = 1, \dots, N$

$\Rightarrow$ we still need to perform $(m-1) \cdot N$ integrals to get the joint density given on page 21!

- weight function = Meijer G-function (gen. of hypergeom. fct)

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \oint_L ds\, z^s\, \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)}$$

where the path L in $\mathbb{C}$ depends on the poles of the Γ's (see e.g. NIST handbook)

Theorem of Springer and Thompson (1970): (scalar case)
The product of m random variables with Gauss' (or Γ or β) distribution is distributed according to a Meijer G-function.

Examples: $m = 1$: $G^{1\,0}_{0\,1}\left(\begin{matrix} - \\ 0 \end{matrix} \middle| x^2 \right) = e^{-x^2}$

$m = 2$: $G^{2\,0}_{0\,2}\left(\begin{matrix} - \\ 0, 0 \end{matrix} \middle| x^2 \right) = 2 K_0(2x) = \int_0^\infty \frac{dt}{t}\, e^{-t - \frac{x^2}{t}}$