Summary Lecture 7

task: joint density of complex ev of $Y_m = J_1 \cdots J_m$, J_j indep Ginibre

- 1st step taken: generalised Schur decomposition

$$J_j = U_j (Z_j + T_j) U_{j+1}^{\dagger} \qquad j = 1, \dots, m, \quad U_{m+1} = U_1$$

leads to the Jacobian for the change of variables from

$J_j \to \{U_j, Z_j, T_j\}$ in terms of the complex ev z_i in

$$\mathrm{diag}(z_1, \dots, z_N) = Z = Z_1 \cdots Z_m \quad : \quad \sim |\Delta_N(\{z\})|^2 = \prod_{j>i}^{N} |z_j - z_i|^2$$

→ expression on p. 25

- today: 2nd step: computation of the weight = Meijer G-function

For the partition function

$$G_N^{(m)} \propto \prod_{j=1}^{m} \int [dY_j]\, e^{-\mathrm{Tr}(Y_j Y_j^\dagger)}$$

$$= \prod_{j=1}^{m} \left\{ \int_{\text{Haar}} [dU_j] \int [dD_j]\, e^{-\mathrm{Tr}(D_j D_j^\dagger)} \right\} \int_{\mathbb{C}} dz_1^{(1)} \cdots dz_N^{(1)} \cdot e^{-\sum_{i=1}^{N} |z_i^{(1)}|^2} \cdots$$

this gives a const.

$$\cdot \int_{\mathbb{C}} dz_1^{(m)} \cdots dz_N^{(m)}\, e^{-\sum_{i=1}^{N} |z_i^{(m)}|^2} \left|\Delta_N(\{z\})\right|^2 \prod_{j=1}^{N} \delta^{(2)}\!\left(z_j - z_j^{(1)} \cdots z_j^{(m)}\right)$$

this contains already the complex ev $z_i = z_i^{(1)} \cdots z_i^{(m)}$ of Y_m, $i = 1, \dots, N$

$\Rightarrow$ we still need to perform $(m-1)\cdot N$ integrals to get the joint density given on page 21!

- weight function = Meijer G-function (gen. of hypergeom. fct.)

$$G^{m,n}_{p,q}\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right) = \frac{1}{2\pi i} \oint_L ds\, z^s \, \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)}$$

where the path L in $\mathbb{C}$ depends on the poles of the Γ's (see e.g. NIST handbook)

Theorem of Springer and Thompson (1970) SIAM: (scalar case)

The product of m random variables with Gauss' (or Γ or β) distribution is distributed according to a Meijer G-function.

Examples: $m = 1$: $G^{1,0}_{0,1}\left(\begin{matrix} - \\ 0 \end{matrix} \middle| x^2\right) = e^{-x^2}$

$m = 2$: $G^{2,0}_{0,2}\left(\begin{matrix} - \\ 0, 0 \end{matrix} \middle| x^2\right) = 2 K_0(2x) = \int_0^\infty \frac{dt}{t}\, e^{-t - \frac{x^2}{t}}$

Exercise 19: show that this results from 2 Gauss' random variables

◦ properties of the weight function $w_m(z)$:

– for a single ev, e.g. z_1 we need to consider

$$w_m(z_1) = \int_{\mathbb{C}} dz_1^{(1)} \dots \int_{\mathbb{C}} dz_1^{(m)} \, e^{-|z_1^{(1)}|^2 - \dots - |z_1^{(m)}|^2} \delta^{(2)}(z_1 - z_1^{(1)} \cdots z_1^{(m)})$$

↳ only depends on $|z_1|$

* factorising moments: (drop index)

$$\int_{\mathbb{C}} d^2 z \, |z|^{2k} w_m(z) = \int_{\mathbb{C}} d^2 z \int_{\mathbb{C}} d^2 z^{(1)} e^{-|z^{(1)}|^2} \dots \int_{\mathbb{C}} d^2 z^{(m)} e^{-|z^{(m)}|^2} \underbrace{|z|^{2k}}_{|z^{(1)}|^{2k} \dots |z^{(m)}|^{2k}} \delta^{(2)}(z - z^{(1)} \cdots z^{(m)})$$

$$= \left(\int_{\mathbb{C}} d^2 u \, |u|^{2k} e^{-|u|^2} \right)^m = (\pi k!)^m$$

* recursion relation

$$\int_{\mathbb{C}} d^2 z \, w(z) = \int_0^{2\pi} d\phi \int_0^{\infty} dr \, (2\pi)^{m-1} \int_0^{\infty} dr_1 r_1 \dots \int_0^{\infty} dr_{m-1} r_{m-1} \frac{r}{(r_1 \cdots r_{m-1})^2}$$

$$e^{-r_1^2} \dots e^{-r_{m-1}^2} \cdot e^{-\frac{r^2}{(r_1 \cdots r_{m-1})^2}}$$

using $z = r e^{i\phi}$ $\quad \delta(f(x)) = \frac{1}{|f'(x_0)|} \delta(x - x_0)$ for single zero of $f(x)$

and using the $\delta^{(2)}$-function to perform the integral over z_m

$$\Rightarrow w_m(z) = w_m(|z|) = (2\pi)^{m-1} \int_0^{\infty} \frac{dr_1}{r_1} \dots \int_0^{\infty} \frac{dr_{m-1}}{r_{m-1}} \, e^{-\frac{|z|^2}{(r_1 \cdots r_{m-1})^2} - \sum_{j=1}^{m-1} r_j^2}$$

$m > 1$

i.e. ∃ recursive relation

$$w_{m+1}(z) = 2\pi \int_0^{\infty} \frac{dr}{r} \, w_m\left(\frac{z}{r}\right) e^{-r^2} \qquad \text{for } m > 0$$

with initial condition $w_{m=1}(z) = e^{-|z|^2}$

→ solution of the recurrence relation: Mellin transform

$$\boxed{\mathcal{M}\{f(r)\}(s) = \int_0^\infty dr\, r^{s-1} f(r) = M(s)}$$

here: consider $\omega_m = \omega_m(|z|)$ depending only on $|z|$ (2) as a function of $R = |z|^2$

so $\Omega_m(R) = \omega_m(\sqrt{R})$ with

$$\mathcal{M}\{\Omega_m(R)\}(s) = \int_0^\infty dR\, R^{s-1}\,\Omega_m(R) = M_m(s)$$

⇒ recursion $\Omega_{m+1}(R) = \omega_{m+1}(\sqrt{R}) = \pi\, \cancel{2}\,\frac{1}{\cancel{2}} \int_0^\infty \frac{d\varrho}{\varrho}\, \omega_m\!\left(\frac{\sqrt{R}}{\sqrt{\varrho}}\right) e^{-\varrho} = \pi \int_0^\infty \frac{d\varrho}{\varrho}\, \Omega_m\!\left(\frac{R}{\varrho}\right) e^{-\varrho}$

substitute $\varrho = r^2$

which is an integral relation factorises

$$\Rightarrow \boxed{M_{m+1}(s)} = \int_0^\infty dR\, R^{s-1}\,\Omega_{m+1}(R) = \int_0^\infty dR\, R^{s-1}\, \pi \int_0^\infty \frac{d\varrho}{\varrho}\, \Omega_m\!\left(\frac{R}{\varrho}\right) e^{-\varrho}$$

substitute $R' = \frac{R}{\varrho}$

$$= \pi \int_0^\infty \frac{d\varrho}{\varrho}\, e^{-\varrho} \int_0^\infty \frac{dR'}{R'}\, R'^{\,s-1}\, \varrho^{s}\, \Omega_m(R') = \boxed{\pi\,\Gamma(s)\, M_m(s)}$$

with initial condition $M_1(s) = \int_0^\infty dR\, R^{s-1} \underbrace{\Omega_1(R)}_{\omega_1(\sqrt{R}) = e^{-R}} = \Gamma(s)$

$$\Rightarrow \boxed{M_m(s) = \pi^{m-1}\,\Gamma(s)^m}$$, solve for $\Omega_{m+1}(R)$ by

inverse Mellin transform $\mathcal{M}^{-1}\{M(s)\}(r) = \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i}\, r^{-s} M(s)$

$$\Rightarrow \boxed{\omega_m(z) = \Omega_m(|z|^2)} = \int_{c-i\infty} \frac{ds}{2\pi i}\, |z|^{-2s}\, \pi^{m-1}\, \Gamma(s)^m = \pi^{m-1}\, G^{m0}_{0m}\!\left(\begin{matrix} - \\ \underbrace{0,\dots,0}_{m} \end{matrix} \,\middle|\, |z|^2\right)$$

Compared to p. 25 with the def of Meijer G, after $s \to -s$ for $b_j = 0\ \forall j = 1,\dots,m$ and a_j's absent.

Complex eigenvalue correlation functions for products of matrices

- based on the fact that the weight $w_m(z) = w_m(|z|)$ we can immediately read off the <u>orthogonal polynomials</u>

$$P_n(z) = z^n$$ to be monic (see page 5)

$\Rightarrow$ from the knowledge of the moments we have the norms

$$\int_{\mathbb{C}} d^2z\, w_m(z)\, z^k \bar{z}^l = \delta_{kl} h_k$$

with $h_k = (\pi k!)^m$ and consequently the

<u>Kernel of OP</u> $K_N^{(m)}(z,\bar{u}) = \sum_{l=0}^{N-1} \frac{z^l \bar{u}^l}{(\pi l!)^m}$ and the

k point corr. funct. $R_k(z_1,\dots,z_k) = \prod_{l=1}^{k} w_m(z_l)\, \det\left[K_N^{(m)}(z_i,\bar{z}_j)\right]_{i,j=1}^{k}$

e.g. for k=1 the <u>spectral density</u>

$$\boxed{R_1(z) = \pi^{m-1}\, G^{m0}_{0m}\left({- \atop 0,\dots,0}\,\middle|\, |z|^2\right) \sum_{l=0}^{N-1} \frac{|z|^{2l}}{\pi^m (l!)^m}}$$

<u>Exercise 20:</u> Show that for products of induced Ginibre matrices

$$P(J_i) \sim \det(J_i J_i^\dagger)^{\nu_i}\, e^{-\pi \operatorname{Tr} J_i J_i^\dagger}, \quad \nu_i > -1$$

we obtain $w_m(z) = \pi^{m-1} G^{m0}_{0m}\left({0 \atop \nu_1,\dots,\nu_m}\,\middle|\, |z|^2\right)$, $h_k = \pi^m \prod_{j=1}^{m} \Gamma(j+1+\nu_j)$

<u>Exercise 21:</u> Show that if we allow these to have different variances,

$e^{-t_i \pi \operatorname{Tr} J_i J_i^\dagger}$ we obtain $w_m(z;\{t\}) = \frac{\pi^{m-1}}{\prod_{j=1}^{m} t_j^{\nu_j}}\, G^{m0}_{0m}\left({- \atop \nu_1,\dots,\nu_m}\,\middle|\, t_1\cdots t_m |z|^2\right)$

which again only depends on $|z|^2$.

Distribution of radii and hole probab. for products of matrices [1211.1576]

- due to $\omega_m(z) = \omega_m(|z|=r)$ we can immediately read off the distribution of radii $r_j = |z_j|$ of the complex ev of Y_m

p. 10: $$\underline{P_N(r_1,\dots,r_N) = \prod_{j=1}^{N} \omega_m(r_j)\ \mathrm{Per}\left[r_\ell^{2j-1}\right]_{j,\ell=1}^{N}}$$

and the Theorem of Kostlan from p. 10 generalises to the statement that the random variables R_k^2 have the density

$$g_k^{(m)}(y) = y^{k-1}\,\omega_m(\sqrt{y}) = \pi^{m-1}\, y^{k-1}\, G^{m\,0}_{0\,m}\left(\overline{0,\dots,0}\,\middle|\, y\right)$$
$$= \pi^{m-1}\, G^{m\,0}_{0\,m}\left(\overline{k-1,\dots,k-1}\,\middle|\, y\right)$$

This is due to Exercise 22: for general Meijer G-functions it holds

$$x^k\, G^{mn}_{pq}\left(\begin{matrix}a_1\dots a_p\\ b_1\dots b_q\end{matrix}\,\middle|\, x\right) = G^{mn}_{pq}\left(\begin{matrix}a_1+k,\dots,a_p+k\\ b_1+k,\dots,b_q+k\end{matrix}\,\middle|\, x\right)$$

- the computation of the k-th gap probab. and individual (radial) ev distrib can be computed as for Ginibre, see p. 16.

- the hole probability $E_0(r)$, that all ev have a radius $\geq r$ can be computed most explicitly & using p. 17 we have

$$\boxed{E_0(r) = \prod_{j=1}^{N}(1-f_j)} \quad \text{with}$$

$$1-f_j = \frac{\int_0^\infty ds\, s^{j-1}\omega_m(\sqrt{s}) - \int_0^{r^2} ds\, s^{j-1}\omega_m(\sqrt{s})}{h_{j-1}} = \frac{\int_{r^2}^\infty ds\, s^{j-1}\, G^{m\,0}_{0\,m}(\overline{0\dots0}\,|\,s)}{((j-1)!)^m}$$

$$\Leftrightarrow \boxed{1-f_j = \frac{G^{m+1\,0}_{1\,,m+1}\left(\begin{matrix}1\\0,j,\dots,j\end{matrix}\,\middle|\, r^2\right)}{\Gamma(j)^m}}$$ using the integral identity for Meijer G-funct.

$$\int_1^\infty dx\, x^{-s}(x-1)^{\sigma-1}\, G^{mn}_{pq}\left(\begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q\end{matrix}\middle|\alpha x\right) = \Gamma(\sigma)\, G^{m+1,n}_{p+1,q+1}\left(\begin{matrix} a_1 \dots a_p, s \\ s-\sigma, b_1 \dots, b_q\end{matrix}\middle|\alpha\right)$$

[there are many further integral identities for these functions, see Gradshteyn-Ryzhik or e.g. 1307.7560].

Example for the product of $\underline{m=2}$ coupled random matrices :

- so far we have encountered only 1 example for non-trivial, that is non-monic OP on $\mathbb{C}$ $P_n(z) \neq z^n$:
  the elliptic Ginibre ensemble (see p. 3) having Hermite polynomials $He_n\left(\frac{z}{\sqrt{2\tau}}\right)$ (see p. 7)

Q: are there other examples? Yes [hep-th/0403131, hep-th/0507150, we use this convention → 1003.4222] Laguerre OP !

$$\boxed{P(J_1, J_2) \propto \exp\left[-\frac{1}{1-\tau}\operatorname{Tr}\left(J_1J_1^\dagger + J_2 J_2^\dagger - \tau J_1 J_2 - \tau J_2^\dagger J_1^\dagger\right)\right]}$$

with $\tau \in [0,1)$, $J_1, J_2^\dagger$ $N\times(N+\nu)$ complex non-Hermitian matrices

the complex ev of $D = \begin{pmatrix} 0 & J_1 \\ J_2 & 0\end{pmatrix} \Leftrightarrow \det(z - D) = \det(z^2 \mathbb{1}_N - J_1 J_2) = 0$

have a joint density

$$\boxed{P_N(z_1, \dots, z_N) = \prod_{k=1}^N w_\tau(z_k^2)\ \left|\Delta_N(\{z^2\})\right|^2}$$

with $w_\tau(z) = |z|^{\nu+1} \exp\left[\frac{\tau}{1-\tau^2}(z^2 + \bar{z}^2)\right] K_\nu\left(\frac{2|z|^2}{1-\tau^2}\right)$

* for $\tau = 0$ (and $\nu=0$) we are back to $m=2$ indep matrices (and ev of $J_1 J_2$) see p. 23. It holds

$$\boxed{\int_{\mathbb{C}} d^2z\, |z|^\nu e^{b\operatorname{Re}(z)} K_\nu(a|z|)\, L_j^\nu(cz)\, L_k^\nu(c\bar{z}) = \delta_{jk} h_k, \quad a > b > 0, \quad c = \frac{a^2-b^2}{2b}}$$

with $h_j^\nu = \frac{\pi (j+\nu)!}{a\, j!}\left(\frac{a}{b}\right)^{2j}\left(\frac{2a}{a^2-b^2}\right)^{\nu+1}$