Summary Lecture 8

- we have completed the (Jacobian and) weight function for the complex ev $z_{j=1,\dots,N}$ of the product $Y_m = J_1 \cdots J_m$ of m independent Ginibre matrices: $C_N = \text{const} \prod_{j=1}^{N} \int_{\mathbb{C}} d^2 z_j \, w(z_j) \, |\Delta_N(z)|^2$

multi. integral representation $(m>1)$

$$w_m(z) = (2\pi)^{m-1} \int_0^\infty \frac{dr_1}{r_1} \cdots \int_0^\infty \frac{dr_{m-1}}{r_{m-1}} \, e^{-\sum_{j=1}^{m-1} r_j^2 - \frac{|z|^2}{(r_1 \cdots r_{m-1})^2}}$$

depends on $|z|$ only — for $m>1$

$\Rightarrow$ recursion

$$w_{m+1}(z) = 2\pi \int_0^\infty \frac{dr}{r} \, w_m\left(\frac{z}{r}\right) e^{-r^2},$$ solution via

Mellin-transform $\mathcal{M}f(s) = \int_0^\infty dr \, r^{s-1} f(r)$

$\Rightarrow$ Single integral rep

$$w_m(z) = \pi^{m-1} \int_{c-i\infty}^{c+i\infty} \frac{ds}{2\pi i} |z|^{-2s} \Gamma(s)^m = \pi^{m-1} G^{m\,0}_{0\,m}\left({- \atop 0 \dots 0} \middle| |z|^2 \right)$$

ν_j's

* this can be generalised to products of induced Ginibre J_j's

- applying the machinery of OP on $\mathbb{C}$ we computed all complex ev k-point correl funct & gap probabilities

today: • example for $m=2$ product of 2 coupled random matrices

• start with large-N limit

$$\int_1^\infty dx\, x^{-s} (x-1)^{\sigma-1}\, G^{mn}_{pq}\left(\begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q\end{matrix}\Big| \alpha x\right) = \Gamma(\sigma)\, G^{m+1,n}_{p+1,q+1}\left(\begin{matrix} a_1 \dots a_p, s \\ s-\sigma, b_1, \dots, b_q\end{matrix}\Big| \alpha\right)$$

[there are many further integral identities for these functions, see Gradshteyn-Ryzhik or e.g. 1307.7560].

Example for the product of $\underline{m=2}$ coupled random matrices :

- so far we have encountered only 1 example for non-trivial, that is non-monic OP on $\mathbb{C}$ $P_n(z) \neq z^n$:
  the elliptic Ginibre ensemble (see p. 3) having Hermite polynomials $He_n\left(\frac{z}{\sqrt{2\tau}}\right)$ (see p. 7)

Q : are there other examples? Yes [hep-th/0403131, hep-th/0507150, we use this convention → 1003.4222] Laguerre OP !

$$\boxed{P(J_1, J_2) \propto \exp\left[-\frac{1}{1-\tau} \mathrm{Tr}\left(J_1 J_1^\dagger + J_2 J_2^\dagger - \tau\left(J_1 J_2 + J_2^\dagger J_1^\dagger\right)\right)\right]}$$

with $\tau \in [0,1)$, $J_1, J_2^\dagger$ $N \times (N+\nu)$ complex non-Hermitian matrices

the complex ev of $D = \begin{pmatrix} 0 & J_1 \\ J_2 & 0\end{pmatrix} \Leftrightarrow \det(z - D) = \det(z^2 \mathbb{1}_N - J_1 J_2) = 0$

have a joint density

$$\boxed{P_N(z_1, \dots, z_N) = \prod_{k=1}^N w_\tau(z_k^2)\ |\Delta_N(\{z^2\})|^2}$$

with $w_\tau(z) = |z|^{\nu+1} \exp\left[\frac{\tau}{1-\tau^2}(z+\bar z)\right] K_\nu\left(\frac{2|z|}{1-\tau^2}\right)$

---

* for $\tau = 0$ (and $\nu = 0$) we are back to $m=2$ indep matrices (and ev* of $J_1 J_2$) see p. 23. It holds

$$\boxed{\int_{\mathbb{C}} d^2 z\, |z|^\nu e^{b \mathrm{Re}(z)} K_\nu(a|z|)\, L_j^\nu(cz)\, L_k^\nu(c\bar z) = \delta_{jk} h_k , \quad a > b > 0,\ c = \frac{a^2-b^2}{2b}}$$

with $h_j^\nu = \frac{\pi (j+\nu)!}{a\, j!} \left(\frac{a}{b}\right)^{2j} \left(\frac{2a}{a^2-b^2}\right)^{\nu+1}$

Consider 2 indep Ginibre matrices, $X_{1,2}$, $N\times N$ for simplicity $\to$ L_i

$$P(X_j) = c\, e^{-2\operatorname{Tr} X_j X_j^\dagger} \quad \text{for } j=1,2.$$

$$c_N = \int_{\mathbb{C}^{N\times N}} [dX_1] \int_{\mathbb{C}^{N\times N}} [dX_2]\; e^{-2\operatorname{Tr} X_1 X_1^\dagger - 2\operatorname{Tr} X_2 X_2^\dagger}$$

Suppose that we are interested in the complex eigenvalues of the matrix

$$D = \begin{pmatrix} 0 & \sqrt{1+\tau}\, X_1 + \sqrt{1-\tau}\, X_2 \\ \sqrt{1+\tau}\, X_1^\dagger - \sqrt{1-\tau}\, X_2^\dagger & 0 \end{pmatrix}, \quad \tau \in [0,1]$$

def
$$\boxed{\begin{aligned} J_1 &= \sqrt{1+\tau}\, X_1 + \sqrt{1-\tau}\, X_2 \\ J_2 &= \sqrt{1+\tau}\, X_1^\dagger - \sqrt{1-\tau}\, X_2^\dagger \end{aligned}}$$
; clearly $J_1^\dagger \neq J_2$ for $\tau<1$
$J_1^\dagger = J_2$ for $\tau = 1$

$\Rightarrow$ ev of D: $0 = \det(z\mathbb{1} - D) = \begin{vmatrix} z & J_1 \\ J_2 & z \end{vmatrix} = \det(z^2 - J_1 J_2)$

are the ev of the product of $m=2$ matrices $Y_2 = J_1 J_2$,

<u>unless $\tau = 1$</u>, then the D ev are the ev of $\underbrace{J_1 J_1^\dagger}_{\text{pos. definite} \Rightarrow z \to \text{real}}$ of a single J_1

Q: What is the joint density of matrix elements of J_1 and J_2?

<u>Exercise 2.3:</u> (change variables $X_{1,2} \to J_{1,2}$)

$$P(J_1, J_2) \propto \exp\left[-\frac{1}{1-\tau} \operatorname{Tr}\left(J_1 J_1^\dagger + J_2 J_2^\dagger - \tau (J_1 J_2 + J_2^\dagger J_1^\dagger)\right)\right]$$

these 2 matrices are coupled, unless when $\tau = 0$, as the sum of 2 indep Gauß matrices with same variance are again Gauß! (with different variance)

* the computation of the joint density of complex ev $z_1, \dots, z_N$ of Y_2 is completely analogous to 2 indep matrices, pp. 23 - 26.

Why: $$\boxed{\exp\left[\frac{\tau}{1-\tau}\mathrm{Tr}(J_1J_2 + J_1^\dagger J_2^\dagger)\right] = \exp\left[\frac{\tau}{1-\tau}\left(\sum_{j=1}^{N}(z_j^2+\bar z_j^2)\right)\right]}$$ D-ev

already contains the product matrix $Y_2 = J_1 J_2 = U(Z+T)U^\dagger$

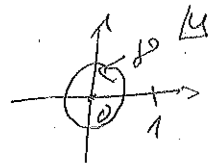
exception: when J_1, J_2: $N\times(N+\nu)$ rectangular, the Jacobian of the generalised Schur decomp reads $\prod_{j=1}^{N}|z_j|^{2\nu}|\Delta_N(z^2)|^2$

$\Rightarrow K_{\nu}$ instead of K_0

(Exercise 24: show this, starting from p. 26)

using $K_\nu(x) = \frac{1}{2}\left(\frac{x}{2}\right)^\nu \int_0^\infty \frac{dt}{t^{\nu+1}} e^{-t-\frac{x^2}{4t}}$

* the orthogonality of the Laguerre polynomials wrt the weight on page 30 can be shown using induction and the integral rep:

$$L_j^{(\nu)}(x) = \oint_\gamma \frac{du}{2\pi i}\frac{e^{-\frac{xu}{1-u}}}{(1-u)^{\nu+1}u^{j+1}},$$ 

(Exercise 25)

$\Rightarrow$ the resulting kernel of OP is given by (check!)

$$\boxed{K_N(z,\bar v) = \frac{8}{\pi(1-\tau^2)}\cancel{\left(w_\tau(z^2)w_\tau(\bar v^2)\right)^{\frac{1}{2}}}\sum_{k=0}^{N-1}\frac{\tau^{2k}k!}{(k+\nu)!}L_k^{(\nu)}\left(\frac{z^2}{\tau}\right)L_k^{(\nu)}\left(\frac{\bar v^2}{\tau}\right)}$$

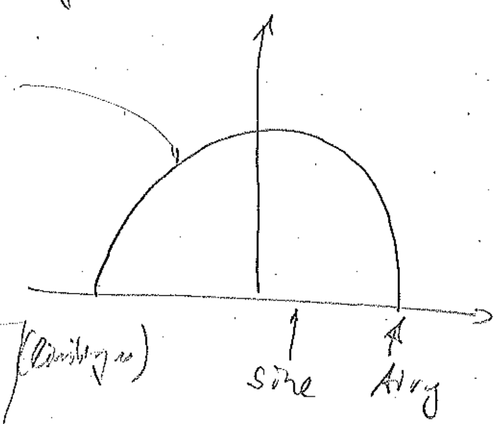
- limit $\tau \to 0$: $L_k^{(\nu)}\left(\frac{z^2}{\tau}\right) \sim (-)^k\frac{z^{2k}}{\tau^k k!} + O(\tau^{-k+1})$

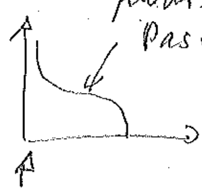
leads back to p. 28 kernel of $n=2$ indep matrices $\sim \sum_{k=0}^{N-1}\frac{z^{2k}\bar v^{2k}}{(k!)^2}$ (in $z^2, \bar v^2$, $\nu=0$)

* the k-point complex ev correlation functions follow as for the elliptic Ginibre ensemble, see p 9. However, as there the gap probab. are no longer simple, due to the lack of rotational invariance.

The large-N limit in non-Hermitian RMT

* for Hermitian RMT we distinguish e.g. for the GUE
  - the <u>global regime</u>: semi-circle
    (= macroscopic)
  - the <u>local regime</u> (microscopic),
    depends on the location of the spectrum: $\boxed{\text{Sine- (bulk) and Airy-kernel}}$ (edge)
* for the chiral GUE there is a third local regime: $\boxed{\text{Bessel-kernel}}$ (origin = hard edge)





• these result from <u>different rescalings</u>:

$$P(H) = e^{-\mathrm{Tr}H^2} \quad \rightarrow \quad \text{support} \sim [-\sqrt{N}, \sqrt{N}] \quad \text{global}$$

or N $\qquad [-1, 1]$

on the <u>local</u> fluctuations by zooming into $z = z_0 + N^{\delta} w$

How does this work for non-Hermitian (e.g. Ginibre)?

- <u>global density</u> (& support) on $\mathbb{C}$  circular law
- $\exists$ different <u>local limits</u> for Ginibre as well?
  Yes: <u>bulk and edge</u>, and for product $\exists$ different origin limit.