Summary Lecture 9

* We have seen an example for the product of 2 coupled random matrices J_1, J_2, $Y_2 = J_1 J_2$

$$P(J_1, J_2) \sim \exp\left[-\frac{1}{1-\tau} \operatorname{Tr}\left(J_1 J_1^\dagger + J_2 J_2^\dagger - \tau\left(J_1 J_2 + J_2^\dagger J_1^\dagger\right)\right)\right]$$

that can be solved using Laguerre polynomials on $\mathbb{C}$

with Kernel $$K_N(z, \bar{v}) \sim \sum_{l=0}^{N-1} \frac{\tau^{2l} l!}{(l+\nu)!} L_l^{(\nu)}\left(\frac{z^2}{\tau}\right) L_l^{(\nu)}\left(\frac{\bar{v}^2}{\tau}\right)$$

and weight $$w(z^2) = |z|^{2\nu+2} e^{\frac{\tau}{1-\tau}(z^2+\bar{z}^2)} K_\nu\left(\frac{2|z|^2}{1-\tau}\right)$$

$$\sim G^{20}_{02}\left(\overline{\nu, 0} \,\middle|\, \frac{|z|^4}{1-\tau}\right)$$

* we have started to investigate the large-N limit distinguishing global / local statistics as for Hermitian RMT

• The large-N limit(s) of the Ginibre ensemble ($\nu=0$ for simplicity)

k-point correlation functions: p. 6

$$R_k(z_1 \dots z_k) = \prod_{j=1}^{k} \omega(z_j) \det_{1\le l,m\le k}\left[K_N(z_l,\bar{z}_m)\right]$$

$$= \det\Big[\underbrace{\omega(z_l)^{\frac{1}{2}}\omega(\bar{z}_m)^{\frac{1}{2}} K_N(z_l,\bar{z}_m)}_{\mathcal{K}_N(z_l,\bar{z}_m)}\Big] \quad , \omega(z)=e^{-|z|^2}$$

$\Rightarrow$ we need to study the limit of weights. kernel

* "naive limit" for Ginibre, p. 7

$$\lim_{N\to\infty} e^{-\frac{1}{2}|z|^2} e^{-\frac{1}{2}|u|^2} \sum_{l=0}^{N-1} \frac{(z\bar{u})^l}{\pi\, l!} = \boxed{e^{-\frac{1}{2}|z|^2-\frac{1}{2}|u|^2} \frac{1}{\pi} e^{z\bar{u}} = \mathcal{K}_{Gin}(z,\bar{u})}$$

$|z|,|u| = O(1)$, N indep!

limiting Ginibre-kernel

$$\Rightarrow \boxed{\lim_{N\to\infty} R_k(z_1 \dots z_k) = \prod_{j=1}^{k} \frac{e^{-|z_j|^2}}{\pi^k} \det_{1\le l,m\le k}\left[e^{z_l\bar{z}_m}\right]}$$

e.g. $k=1$ spectral density $R_1(z) = \mathcal{K}_{Gin}(z,\bar{z}) = e^{-|z|^2}\frac{1}{\pi}e^{z\bar{z}} = \frac{1}{\pi}$

constant density

$k=2$ two-point function: $\underline{R_2(z_1,z_2)} = e^{-|z_1|^2-|z_2|^2}\frac{1}{\pi^2}\begin{vmatrix} e^{z_1\bar{z}_1} & e^{z_1\bar{z}_2} \\ e^{z_2\bar{z}_1} & e^{z_2\bar{z}_2}\end{vmatrix}$

$$= \frac{e^{-|z_1|^2-|z_2|^2}}{\pi^2}\left(e^{|z_1|^2+|z_2|^2} - e^{z_1\bar{z}_2+z_2\bar{z}_1}\right)$$

$$= \frac{1}{\pi^2}\Big(1 - \underbrace{e^{-(z_1-z_2)(\bar{z}_1-\bar{z}_2)}}_{|z_1-z_2|^2}\Big)$$

only depends on the difference z_1-z_2, both $|z_1|,|z_2| = O(1)$

factorised $R_1(z_1)R_1(z_2)$ part

Are these local or global correlation functions?

→ For this we need to find out the support of the limiting density on a global scale:

recall p. 7: $K_N(z,\bar u) = \sum_{\ell=0}^{N-1} \frac{(z\bar u)^\ell}{\pi \ell!} = \frac{e^{z\bar u}}{\pi} \frac{\Gamma(N, z\bar u)}{\Gamma(N)}$,

where $\Gamma(N,x) = \int_x^\infty dt\, t^{N-1} e^{-t}$ is the incomplete Gamma function

with $\Gamma(N,0) = \Gamma(N) = (N-1)!$ asymptotic from Stirling formula

$$\Rightarrow \boxed{R_1(z) = e^{-|z|^2} e^{|z|^2} \frac{1}{\pi} \frac{\Gamma(N,|z|^2)}{\Gamma(N)} = \frac{1}{\pi}\frac{\Gamma(N,|z|^2)}{\Gamma(N)}}$$ and it is well-known

that $\boxed{\lim_{N\to\infty} R_1(z) = \lim_{N\to\infty} \frac{1}{\pi}\frac{\Gamma(N,|z|^2)}{\Gamma(N)}} \approx \boxed{\frac{1}{\pi}\Theta(\sqrt{N} - |z|)}$ with $\int d^2z\, R_1(z) = N$

Heaviside function

Exercise: show this using a saddle point approx. to the integral for $\Gamma(N,x)$, around $x=\sqrt{N}$

* This is the circular law, that is for $|z| \le \sqrt{N}$ the density is constant, so to get a compact support we need to rescale matrix elements or complex ev as in the GUE

$$z_j \to \sqrt{N}\, \hat z_j \qquad \forall j = 1,\dots,N$$

$\Rightarrow P(J) \sim e^{-N \operatorname{Tr} JJ^\dagger}$ has compact support, with

$$\rho_1(z) = \lim_{N\to\infty} R_1(\hat z) = \lim_{N\to\infty} \frac{\Gamma(N, N|\hat z|^2)}{\pi\, \Gamma(N)} = \frac{1}{\pi}\Theta(\sqrt{N} - \sqrt{N}|\hat z|) = \boxed{\frac{1}{\pi}\Theta(1-|\hat z|)}$$

which is now normalised to unity $\int d^2z\, \rho_1(z) = 1$

(→ translation?)

⇒ the previous "naive limit" is a microscopic limit close to the origin as in $\hat z, \hat u$: $\mathcal{K}_N(\hat z, \hat u) = e^{-\frac{N}{2}(|\hat z|^2 + |\hat u|^2)} \sum_{\ell=0}^{N-1} \frac{N(\hat z \bar{\hat u})^\ell}{\pi \ell!}$ needs a rescaling $\hat z \to \frac{1}{\sqrt{N}}\hat z$

* because the density is flat on the unit disc, we may expect that the limiting

Ginibre-Kernel $K_{Gin}(z,\bar{u}) = \frac{1}{\pi} e^{-\frac{|z|^2}{2} - \frac{|u|^2}{2}} e^{z\bar{u}}$

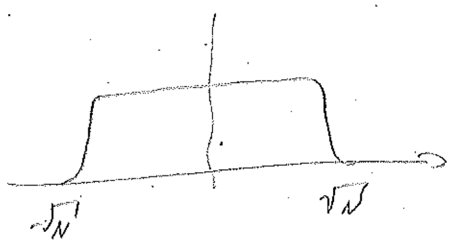
holds everywhere on $|z|<1$, due to translational invariance.

- limiting kernel at the edge: complementary error function erfc

in the original scaling $P(J) \propto e^{-\mathrm{Tr} JJ^\dagger}$ we have seen

that $R_1(z) = \frac{1}{\pi} \frac{\Gamma(N, |z|^2)}{\Gamma(N)} \simeq \frac{1}{\pi} \Theta(\sqrt{N} - |z|)$

so let's zoom into the vicinity of the edge: $z = (\sqrt{N} + x)\, e^{i\varphi}$



$$\Rightarrow \lim_{\substack{N\to\infty \\ x = O(1)}} R_1\left(z = (\sqrt{N}+x)e^{i\varphi}\right) \simeq \frac{1}{2\pi}\, \mathrm{erfc}(\sqrt{2}\, x)$$ Exercise 26

where $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty dt\, e^{-t^2}$

with asymptotics $\boxed{\lim_{x\to-\infty} \mathrm{erfc}(x) = 2, \quad \lim_{x\to+\infty} \mathrm{erfc}(x) = \frac{e^{-x^2}}{\sqrt{\pi}\, x}}$

for the limiting kernel see e.g. 1312.0068, Prop. 8.1

* note that it can be shown, that the eigenvalue largest in modulus is Gumbel distributed

$$\boxed{P(x) = e^{-e^{-x}}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (e^{-x})^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_x^\infty dx_1 \ldots \int_x^\infty dx_n\, e^{-x_1 - \ldots - x_n}$$

like indep. Poisson distrib. random variables ~ degenerate det kernel

see 0808.2608

so far we have: global – circular law

local – Ginibre kernel bulk (origin),

erfc – kernel edge

- $\exists$ other limiting kernels? yes, for products and elliptic Gin
- are these limiting kernels universal, that is do they hold for distributions of matrix element J_{ij} different from Gaussian $e^{-\mathrm{Tr}\, JJ^\dagger}$?

(Ex 2 in Lec: it is difficult to write down convergent $P(J)$ s.that ev $z_1 \ldots z_N$ decouple) Yes, e.g. for Wigner matrices:

J_{ij} indep random variables, s.th. certain (all) moments exist

T. Tao, V. Vu 1206.1893

not much more rigorous results for non-Hermitian RMT (e.g. 1610.06517)

$\rightarrow$ the weak non-Hermiticity limit [Fyodorov, Khoruzhenko, Sommers chao-dyn/9802025]

it interpolates between sine-kernel GUE and Ginibre kernel, as we will show

$\rightarrow$ * Consider the elliptic Ginibre ensemble (p. 7) $z = x + iy$

weight $w_\tau(z) = \exp\left(-\frac{1}{1-\tau^2}\left(\underbrace{|z|^2}_{x^2+y^2} - \frac{\tau}{2}\underbrace{(z^2+\bar{z}^2)}_{2(x^2-y^2)}\right)\right) = \exp\left(-\frac{x^2}{1+\tau} - \frac{y^2}{1-\tau}\right)$

Kernel $K_N(z,\bar{u}) = \sum_{\ell=0}^{N-1} \frac{1}{\ell!}\left(\frac{\tau}{2}\right)^\ell He_\ell\left(\frac{z}{\sqrt{2\tau}}\right) He_\ell\left(\frac{\bar{u}}{\sqrt{2\tau}}\right)$

global limit: one can show (see above reference)

$$\lim_{N\to\infty} \frac{1}{N} R_1(\sqrt{N}z) = \begin{cases} \frac{1}{\pi(1-\tau^2)} & \text{if } \frac{x^2}{(1+\tau)^2} + \frac{y^2}{(1-\tau)^2} \leq 1, \quad z = x+iy \\ 0 & \text{else} \end{cases}$$

elliptic law

Universality of the elliptic Ginibre ensemble:

"naive" limit $\lim_{N\to\infty} K_N(z,\bar{u}) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}\left(\frac{\tau}{2}\right)^{\ell} He_\ell\left(\frac{z}{\sqrt{2\tau}}\right) He_\ell\left(\frac{\bar{u}}{\sqrt{2\tau}}\right)$

Mehler formula $= \frac{1}{\sqrt{1-\tau^2}} \exp\left[\frac{1}{1-\tau^2}\left(z\bar{u} - \frac{\tau}{2}(z^2+\bar{u}^2)\right)\right]$

Exercise 22, check in the limit $\tau\to 0$

= k-point correl. function

$$R_k(z_1,\dots,z_k) \sim \prod_{j=1}^{k} e^{-\frac{1}{1-\tau^2}\left(|z_j|^2 - \frac{\tau}{2}(z_j^2+\bar{z}_j^2)\right)} \det_{1\le \ell,m\le k}\left[\frac{1}{\sqrt{1-\tau^2}}\, e^{\frac{z_\ell\bar{z}_m}{1-\tau^2} - \frac{\frac{\tau}{2}(z_\ell^2+\bar{z}_m^2)}{1-\tau^2}}\right]$$

take out and cancel

$$= \frac{1}{(1-\tau^2)^{\frac{k}{2}}} \prod_{j=1}^{k} e^{-\frac{1}{1-\tau^2}|z_j|^2} \det_{1\le \ell,m\le k}\left[e^{\frac{z_\ell\bar{z}_m}{1-\tau^2}}\right]$$

---

same as for Ginibre kernel, after rescaling $z_\ell \to \frac{z_\ell}{\sqrt{1-\tau^2}}$

= universal!

Weak non-Hermiticity

* we consider simultaneously (= double scaling limit)
  $N\to\infty$ and $\tau\to 1$, the Hermitian limit: $\lim_{\tau\to 1}\frac{1}{\sqrt{\pi(1-\tau)}}e^{-\frac{y^2}{1-\tau}} = \delta(y)$
  such that $\boxed{\lim_{\substack{N\to\infty\\ \tau\to 1}} N(1-\tau^2) = \alpha^2}$ is kept constant

$\Rightarrow$ the global density (Ellipse) collapses to the GUE semicircle
But: the local correlations still extend into the complex plane

* scaling of z_j's:
  - for compact support we need $z_j \to \sqrt{N} z_j$, weight $e^{-\frac{N}{1-\tau^2}\left(|z|^2 - \frac{\tau}{2}(z^2+\bar{z}^2)\right)}$

  then $z_{1,2} = X + \frac{x_{1,2}}{N} + i\frac{y_{1,2}}{N} = X + \frac{\xi_{1,2}}{N}$ ($X\in\mathbb{R}$, $\xi_{1,2}\in\mathbb{C}$, $\frac{1}{N}$ deviations from $\mathbb{R}$

weight function: $\omega(z_1) = \exp\left[-\frac{N}{1+\tau}\left(X+\frac{x_1}{N}\right)^2 - \frac{N}{1-\tau}\left(\frac{y_1}{N}\right)^2\right]$

$$\xrightarrow[\substack{N\to\infty \\ \tau\to 1}]{} \exp\left[-\frac{N}{2}X^2 - Xx_1 - \frac{2y_1^2}{\alpha^2}\right]$$

GUE weight

i.e. $\exists$ remaining non-trivial contrib.

* for the kernel the limit is more tricky, involving $\int$-rep's of the Hermite polyn. and a carefull saddle point analysis (see Lit). We cannot use the Mehler formula here, (non uniform)!

result: $\lim\limits_{\substack{N\to\infty \\ \tau\to 1}} \omega(z_1)^{\frac{1}{2}} \omega(\bar{z}_2)^{\frac{1}{2}} \frac{K_N(z_1,\bar{z}_2)}{N^2} \sim \frac{e^{-\frac{y_1^2}{\alpha^2}-\frac{y_2^2}{\alpha^2}}}{\sqrt{\pi}\,\alpha} \int_0^1 du\, e^{-\frac{\alpha^2 u^2}{2}} \cos(u(\xi_1-\xi_2))$

(here for $X=0$ at the origin, else $\int_0^1 \to \int_0^{\sqrt{1-X^2/2}}$)

this is the kernel $K(\xi_1,\xi_2;\alpha)$ (in form of hep-th/0206086)

Interpolating between Sine-kernel & Ginibre:

GUE limit $\alpha\to 0$: $\lim\limits_{\alpha\to 0} \frac{1}{\sqrt{\pi}\alpha} e^{-\frac{y^2}{\alpha}} = \delta(y)$ collapse to $\mathbb{R}$

and $\int_0^1 du\, 1\cdot\cos(u(x_1-x_2)) = \frac{1}{x_1-x_2}\left[\sin(u(x_1-x_2))\right]_0^1 = \frac{\sin(x_1-x_2)}{x_1-x_2}$

Ginibre limit $\alpha\to\infty$: here the scaling of z_i is different ($\sqrt{N}$ vs N in weak lim)

so we have to rescale ξ_i's with $\alpha\sim\sqrt{N}$

and one can show that $\lim\limits_{\alpha\to\infty} K(\xi_1,\xi_2;\alpha)\Big|_{\frac{\xi_{1,2}}{\alpha}\text{ fixed}} \sim K_{Gin}$