Introduction to Random Matrices	**Exercise Sheet No. 1**	15.10.2018

Exercise 1 - Sum and product of two random variables

1. Suppose we have two independent random variables X and Y which are normally distributed, i.e.
$$X \sim N(\mu_X, \sigma_X^2) \quad \text{and} \quad Y \sim N(\mu_Y, \sigma_Y^2) \ . \tag{1}$$
Show that the sum $Z = X + Y$ is also a normally distributed random variable, i.e.
$$Z \sim N(\mu_Z, \sigma_Z^2) \ , \tag{2}$$
and determine μ_Z and σ_Z^2 in terms of μ_X, μ_Y, σ_X^2 and σ_Y^2.

2. Suppose we have two independent standard random variables X and Y, which are Γ-distributed with probability density functions
$$p_X(z) = \frac{1}{\Gamma(a_X)} z^{a_X - 1} e^{-z} \quad \text{and} \quad p_Y(z) = \frac{1}{\Gamma(a_Y)} z^{a_Y - 1} e^{-z} \ , \tag{3}$$
where $z > 0$, $a_X > 0$ and $a_Y > 0$.
Show that the new random variable given by the product $Z = XY$ must have a probability density function of the form
$$p_Z(z) = c_Z z^{a_Z - 1} K_{a_X - a_Y}(2\sqrt{z}) \ , \tag{4}$$
where $K_n(x)$ denotes the modified Bessel function of the second kind. Furthermore, determine the parameters c_Z, a_Z.

Exercise 2 - Level spacing and Wigner's surmise

Suppose we have two coupled random variables x_1,x_2, with joint probability density function (jpdf)
$$\rho(x_1, x_2) = |x_1 - x_2|^\beta e^{-x_1^2} e^{-x_2^2} \ , \tag{5}$$
where $\beta = 1, 2, 4$ respectively. It turns out that this is the jpdf of 2×2 random matrices with real, complex or quaternionic matrix elements, respectively, with Gaussian distribution.

1. Compute the probability density function (pdf) $P_\beta(s)$ of the spacing $s = x_2 - x_1$ between the two eigenvalues ($x_2 > x_1$) for all three values of β.

2. Rescale the obtained pdf such that its mean and normalization are equal to unity, i.e.
$$\int_0^\infty ds P_\beta(s) = 1 \quad \text{and} \int_0^\infty ds P_\beta(s) s = 1 \tag{6}$$
